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DAVID EUGENE SMITH

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David Eugene Smith died at his home in New York City July 29, 1944, at the age of eighty-four after a long illness. He was born in Cortland, New York, and was fortunate in that his mother took a deep and intelligent interest in his early education. She did not leave this important work entirely to the schools, but early taught him Latin and Greek and other subjects, to such good purpose that he could converse in both of these languages.

He entered the State Normal School at Cortland the year it was opened and later enrolled in Syracuse University from which he was graduated at the age of twenty-one. As an undergraduate he acquired a good knowledge of Hebrew and developed a fondness for the Hebrew Bible. Thus he left college with a mastery of the three most important ancient languages, and a strong inclination towards the arts and humanities.

His father was a lawyer and wanted his son to take up the same profession. Accordingly David Eugene studied law in his father's office after his graduation and was admitted to the bar in 1884. But he had no enthusiasm for the Law, and the same year he accepted an offer of an instructorship in mathematics in the Cortland Normal School which he had attended before he entered college. It does not appear that up to this time mathematics had appealed to him strongly. But he entered upon his new duties with enthusiasm and thoroughness, and after seven years he was appointed Professor of Mathematics in the Michigan State Normal School at Ypsilanti. He later served as Principal of the New York State Normal School at Brockport, and in 1901 he become Professor of Mathematics in Teachers College, Columbia University, a position he held until his retirement in 1926, at which time he was made Professor Emeritus.

In 1887 he married Fannie Taylor. She died in 1928 and in 1940 he married Eva May Luce in collaboration with whom he had written some textbooks and who had been Director of Teachers Training in the Iowa State Teachers College.

From the time he gave up the practice of law and began teaching he was unusually active as a writer of text-books and works on the history of mathematics. Among the latter should be mentioned *History of Modern Mathematics*, *Rara Arithmetica*, *History of Japanese Mathematics*, *Our Debt to Greece and Rome in Mathematics*, *Historical Mathematical Paris*, and, in collaboration with Jekuthiel Ginsburg, *A History of Mathematics in America before 1900*. On the pedagogical side, besides the numerous text-books already referred to, there should be mentioned his *Teaching of Arithmetic*. In 1936 and 1937 he brought out two series of Portraits of Eminent Mathematicians.

He translated René Descartes' *La géométrie* from the Latin and French in collaboration with Marcia Lutham and Klein's *Vorträge über ausgewählte Fragen Elementar-geometrie* under the title *Famous Problems of Elementary Geometry* in collaboration with Professor W. W. Beman. He was active in editorial work for

many years. Thus he was Mathematical Editor of the *New International Encyclopedia* from 1902 to 1916, of Monroe's *Cyclopedia of Education* from 1911 to 1913, the *New Practical Reference Library* in 1912, the *Encyclopedia Britannica* in 1927, and the *National Encyclopedia* in 1933. He also served on the editorial board of this MONTHLY and on that of *Scripta Mathematica*. From 1908 to 1920 he was Vice President of the International Committee on the Teaching of Mathematics, President from 1928 to 1932, and Honorary President thereafter. He was Librarian of the American Mathematical Society and Associate Editor of the *Bulletin* of the Society for eighteen years, and Vice President of the Society in 1932.

In addition to all this activity he gave his time freely and generously to the many problems in connection with the organization of the Mathematical Association of America. He was a charter member of the Association, a member of its board of trustees for a number of years, and one of its early presidents. Shortly after the First World War one of the editors of *Bibliotheca Mathematica* suggested to him the desirability of transferring that journal to this country. After conferences with Professor H. E. Slaught of the University of Chicago and Mrs. Mary Hegeler Carus it was decided it would be better to establish here a series of mathematical monographs for which Mrs. Carus would undertake to furnish the financial support. This is the origin of the Carus Monographs.

His love of art which showed itself in his undergraduate days remained with him to the end and prompted him to make a metrical version of the *Rubâiyât* of Omar Khayyám based on a verbatim translation of Haslinn Hussein. For this work he received a decoration from Rega Khan Pahleri who was Shah of Iran at the time. He was an indefatigable collector of material connected with the early history of mathematics, of fine oriental rugs, and, in general, of things of artistic merit. Some years ago he presented his library and collection of mathematical material to Columbia University, where they are suitably housed along with the collections presented by G. A. Plimpton and S. S. Dale.

We have here an impressive list of activities continued over a long period of years, but it gives only a faint and inadequate picture of David Eugene Smith the man.

"Not on the vulgar mass
Called 'work' must sentence pass."

He was not merely an active worker in more or less narrow fields of scholarship. He was learned in a broad field and his learning was tempered by a delightful sense of humor and a mellow and charming personality. In 1921 he delivered the presidential address before the Mathematical Association of America. It was entitled *Religio Mathematici* and was published in volume 28 of this MONTHLY, as well as separately. It reveals with great clearness the depth and richness of his religious feeling. He was a humanist in the finest sense of the word. Those who saw him often in his class room or in his home and who knew him well came to have a great admiration and a deep affection for him.

CERTAIN GENERATING FUNCTIONS AND ASSOCIATED POLYNOMIALS

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1. Introduction. Let $G(x)$ be analytic at $x=0$ and let

$$G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

We shall study the set of polynomials $g_n(x)$ defined by

$$e^t G(xt) = \sum_{n=0}^{\infty} g_n(x) \frac{t^n}{n!}. \tag{1}$$

The polynomials $g_n(x)$ are said to be generated by the function $e^t G(xt)$. Since it involves no loss in generality of the sets of polynomials so defined, let us take $a_0 \neq 0$. Then $g_0(x) \neq 0$.

Milne-Thomson[1] used a generating function resembling that in (1). A forthcoming paper, *On generating functions of polynomial systems*, by W. C. Brenke[2] has equations (3) and (12) below in common with this paper. Our aims and results are, except in an instance or two, not at all similar to those of the two papers mentioned.

We shall obtain a few simple properties of the $g_n(x)$ in general and then proceed to study a specific set of such polynomials.

Application of the Cauchy product of two series gives us at once

$$e^t G(xt) = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} a_n \frac{x^n t^n}{n!} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_k x^k}{k!(n-k)!} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} C_{n,k} a_k x^k \frac{t^n}{n!}.$$

Hence, using (1),

$$g_n(x) = \sum_{k=0}^{n} C_{n,k} a_k x^k. \tag{2}$$

2. A recurrence relation. Differentiation of both sides of (1) with respect to x yields

$$te^t G'(xt) = \sum_{n=1}^{\infty} g_n'(x) \frac{t^n}{n!},$$

while differentiation with respect to t yields

$$e^t G(xt) + xe^t G'(xt) = \sum_{n=1}^{\infty} g_n(x) \frac{t^{n-1}}{(n-1)!}.$$

From these two equations it follows at once that

$$\sum_{n=0}^{\infty} g_n(x)\frac{t^{n+1}}{n!} + \sum_{n=1}^{\infty} xg_n'(x)\frac{t^n}{n!} = \sum_{n=1}^{\infty} g_n(x)\frac{t^n}{(n-1)!},$$

or

$$\sum_{n=1}^{\infty} g_{n-1}(x)\frac{t^n}{(n-1)!} + \sum_{n=1}^{\infty} xg_n'(x)\frac{t^n}{n!} = \sum_{n=1}^{\infty} g_n(x)\frac{t^n}{(n-1)!}.$$

Therefore, for $n \geqq 1$,

$$ng_{n-1}(x) + xg_n'(x) = ng_n(x).$$

Throughout we shall find it convenient to use the operator $\theta = x(d/dx)$. With it, the above equation appears as

$$ng_{n-1}(x) = (n-\theta)g_n(x). \tag{3}$$

The relation (3) is equivalent to the well-known result

$$\alpha_n'(x) = \alpha_{n-1}(x)$$

for Appell polynomials $\alpha_n(x)$ defined by

$$A(t)e^{xt} = \sum_{n=0}^{\infty} \alpha_n(x)t^n.$$

Everything in this paper can be reworded in terms of Appell polynomials by putting $A(t) \equiv G(t)$ and passing from the $g_n(x)$ to the $\alpha_n(x)$ by means of

$$n!\alpha_n(x) = x^n g_n\left(\frac{1}{x}\right).$$

3. Connection with differential equations. Let $G(x)$ satisfy a linear differential equation with polynomial coefficients. Such an equation can always be written in a certain form which is very well suited to many theoretical discussions. That form is obtained by writing the linear differential operator as a sum of terms each containing the product of a polynomial in θ and a single power of x. That is, any linear differential equation with polynomial coefficients may be written as

$$\sum_{k=0}^{m} x^k F_k(\theta)G(x) = 0, \tag{4}$$

where the $F_k(\theta)$; $k = 0, 1, \cdots, m$, are polynomials in the operator θ. It is wise to keep in mind that the operators x and θ are not commutative. Indeed, $\theta xG(x) = x(\theta+1)G(x)$ and $xF_k(\theta)G(x) = F_k(\theta-1)xG(x)$.

We shall concern ourselves primarily with the case in which $G(x)$ satisfies the simple equation

$$[F_0(\theta) + xF_1(\theta)]G(x) = 0. \tag{5}$$

This is an equation (not the only one) which, in the parlance of the subject of power series solutions, may be solved with a two-term recurrence relation.

Let us now determine from (3) and (5) a linear differential equation satisfied by the polynomial $g_n(x)$. Since the generating function involves $G(xt)$, we rewrite (5) with $z=xt$, $\theta_1=z(d/dz)$, as

$$[F_0(\theta_1) + zF_1(\theta_1)]G(z) = 0.$$

Differentiation of the fundamental relation (1) k times with respect to x yields

$$t^k e^{tx} G^{(k)}(xt) = \sum_{n=0}^{\infty} g_n^{(k)}(x) \frac{t^n}{n!}.$$

Hence

$$e^{tx} x^k t^k G^{(k)}(xt) = \sum_{n=0}^{\infty} x^k g_n^{(k)}(x) \frac{t^n}{n!},$$

or

$$e^{tx}\theta_1(\theta_1 - 1) \cdots (\theta_1 - k + 1)G(z) = \sum_{n=0}^{\infty} \theta(\theta - 1) \cdots (\theta - k + 1)g_n(x) \frac{t^n}{n!}.$$

Now successive eliminations yield

$$e^{tx}\theta_1^k G(z) = \sum_{n=0}^{\infty} \theta^k g_n(x) \frac{t^n}{n!}; \qquad k = 0, 1, 2, \cdots. \tag{6}$$

But, from (5),

$$e^{tx}[F_0(\theta_1) + zF_1(\theta_1)]G(z) = 0.$$

Therefore

$$\sum_{n=0}^{\infty} F_0(\theta)g_n(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} xF_1(\theta)g_n(x) \frac{t^{n+1}}{n!} = 0,$$

or

$$\sum_{n=0}^{\infty} F_0(\theta)g_n(x) \frac{t^n}{n!} + \sum_{n=1}^{\infty} xF_1(\theta)g_{n-1}(x) \frac{t^n}{(n-1)!} = 0.$$

We may conclude that $F_0(\theta)g_0(x)=0$, or $F_0(0)=0$, which is merely a reflection of the fact that $G(x)$ is analytic and not zero at $x=0$.

Next, for $n \geqq 1$,

$$F_0(\theta)g_n(x) + nxF_1(\theta)g_{n-1}(x) = 0. \tag{7}$$

We already knew one relation between $g_n(x)$ and $g_{n-1}(x)$, namely,

$$ng_{n-1}(x) = (n - \theta)g_n(x). \tag{3}$$

The elimination of $g_{n-1}(x)$ from (7) and (3) shows that $g_n(x)$ satisfies the differential equation

$$[F_0(\theta) - xF_1(\theta)(\theta - n)]g_n(x) = 0. \tag{8}$$

4. A less restricted equation for $G(x)$. Suppose that $G(x)$ satisfies the less special equation

$$\sum_{k=0}^{m} x^k F_k(\theta)G(x) = 0. \tag{4}$$

From (6) it follows at once that

$$\sum_{k=0}^{m}\sum_{n=0}^{\infty} x^k F_k(\theta)g_n(x)\frac{t^{n+k}}{n!} = 0,$$

or

$$\sum_{k=0}^{m}\sum_{n=k}^{\infty} x^k F_k(\theta)g_{n-k}(x)\frac{t^n}{(n-k)!} = 0, \tag{9}$$

which is the counterpart of (7).

Now, from (3) above it may be seen that

$$(n-1)g_{n-2}(x) = (n-1-\theta)g_{n-1}(x),$$
$$n(n-1)g_{n-2}(x) = (n-\theta)(n-1-\theta)g_n(x),$$

and, in general,

$$n(n-1)\cdots(n-k+1)g_{n-k}(x) = (n-\theta)(n-1-\theta)\cdots(n-k+1-\theta)g_n(x).$$

Here it is convenient to introduce a common notation which will simplify the writing of several formulas in the remainder of the work. Let us define $(\alpha)_k$ by

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1); \qquad (\alpha)_0 = 1.$$

Then the relation preceding the definition of $(\alpha)_k$ may be written

$$\frac{g_{n-k}(x)}{(n-k)!} = (-1)^k(\theta - n)_k\frac{g_n(x)}{n!}. \tag{10}$$

Next, with the aid of (9),

$$\sum_{k=0}^{m}\sum_{n=k}^{\infty}(-1)^k x^k F_k(\theta)(\theta-n)_k g_n(x)\frac{t^n}{n!} = 0,$$

from which the differential equation,

$$\sum_{k=0}^{m} (-1)^k x^k F_k(\theta)(\theta - n)_k g_n(x) = 0 \tag{11}$$

for the $g_n(x)$ follows for $n \geqq m$.

5. Pure recurrence relation. If (3) is put in the form

$$\theta g_n(x) = n[g_n(x) - g_{n-1}(x)]$$

and considered together with

$$F_0(\theta)g_n(x) + nxF_1(\theta)g_{n-1}(x) = 0, \tag{7}$$

in which F_0 and F_1 are polynomials in θ, it is evident that the operator θ may be eliminated. The result will be a pure recurrence relation expressing $g_n(x)$ in terms of a certain number of $g_k(x)$ with subscripts less than n. The coefficients in this relation will be polynomials at most linear in x. As an example, see equation (23) below. When $G(x)$ satisfies the less simple equation (4), a similar relation may be obtained, but with coefficients of higher degree in x.

6. Two identities. Let us return to the original definitions of $g_n(x)$ and $G(x)$ in section 1. Of course,

$$G(xt) = e^{-t}\sum_{n=0}^{\infty} g_n(x)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} g_n(x)\frac{t^n}{n!}\right).$$

The Cauchy product of the two series on the right leads us at once to

$$G(xt) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n-k}\frac{g_k(x)}{(n-k)!k!}t^n,$$

or

$$\sum_{n=0}^{\infty} a_n\frac{x^n t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n-k}C_{n,k}g_k(x)\frac{t^n}{n!}.$$

Therefore

$$\sum_{k=0}^{n}(-1)^k C_{n,k}g_k(x) = (-1)^n a_n x^n. \tag{12}$$

We shall now obtain another identity involving the $g_k(x)$. From the definition (1) it follows that

$$e^t G(xyt) = \sum_{n=0}^{\infty} g_n(xy)\frac{t^n}{n!},$$

and also

$$e^{vt}G(xyt) = \sum_{n=0}^{\infty} g_n(x)\frac{y^n t^n}{n!}.$$

Now,

$$e^{t}G(xyt) = e^{t(1-y)}e^{yt}G(xyt),$$

hence

$$\sum_{n=0}^{\infty} g_n(xy)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty}\frac{(1-y)^n t^n}{n!}\right)\left(\sum_{n=0}^{\infty} g_n(x)\frac{y^n t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{n,k}y^k(1-y)^{n-k}g_k(x)\frac{t^n}{n!}.$$

Therefore

$$g_n(xy) = \sum_{k=0}^{n} C_{n,k}y^k(1-y)^{n-k}g_k(x). \tag{13}$$

Three special cases of (13) appear to be of particular interest.[3] They are

$$g_n(x) = \sum_{k=0}^{n} C_{n,k}g_k(1)x^k(1-x)^{n-k}, \tag{13a}$$

$$g_n(-x) = \sum_{k=0}^{n}(-1)^k C_{n,k}2^{n-k}g_k(x), \tag{13b}$$

$$g_n\left(\frac{x}{2}\right) = 2^{-n}\sum_{k=0}^{n} C_{n,k}g_k(x). \tag{13c}$$

There is a symbolic notation which is convenient here. In this notation it is agreed that exponents shall be lowered to subscripts on any symbol, such as $g(x)$ in our work, which has been undefined except with subscripts. That is, such a symbol as $\{1-g(x)\}^3$ is to be taken to mean

$$g_0(x) - 3g_1(x) + 3g_2(x) - g_3(x).$$

In order to emphasize the presence of the symbolic notation we shall use $\doteq$ to replace $=$ whenever a symbolic equation is given.

The identities of this section may now be written

$$\{1 - g(x)\}^n \doteq \{-ax\}^n, \tag{12}$$

and

$$g_n(xy) \doteq \{1 - y + yg(x)\}^n. \tag{13}$$

The symbolic notation could have been used to arrive at the results of this section in the following way. First, from

$$e^{t}G(xt) \doteq e^{tg(x)} \quad \text{and} \quad G(xt) \doteq e^{axt},$$

we conclude that

$$e^{t\{g(x)-1\}} \doteq e^{axt},$$

or $$\{g(x) - 1\}^n \doteq \{ax\}^n,$$

which is essentially equation (12). Next, from

$$G(xt) \doteq e^{t\{g(x)-1\}}$$

we see that

$$G(xyt) \doteq e^{yt\{g(x)-1\}}.$$

Hence

$$e^t G(xyt) \doteq e^{t g(xy)} \doteq e^{t\{1-y+yg(x)\}}$$

so that

$$\{g(xy)\}^n \doteq \{1 - y + yg(x)\}^n \tag{13}$$

follows.

7. The Laguerre polynomials. For the particular choice

$$G(x) = J_0(2\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(n!)^2},$$

the polynomials $g_n(x)$ are called Laguerre polynomials[4] and are usually written $L_n(x)$. That is, the $L_n(x)$ may be defined by the well-known generating function

$$e^t J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!}.$$

The function $G(x) = J_0(2\sqrt{x})$ satisfies the differential equation

$$xG''(x) + G'(x) + G(x) = 0,$$

or

$$(\theta^2 + x)G(x) = 0.$$

The material in the preceding sections leads at once to the familiar results

$$L_n(x) = \sum_{k=0}^{n} (-1)^k C_{n,k} \frac{x^k}{k!},$$

$$\theta L_n(x) = xL_n'(x) = n[L_n(x) - L_{n-1}(x)],$$

and

$$[\theta^2 - x(\theta - n)]L_n(x) = 0,$$

or

$$xL_n''(x) + (1 - x)L_n'(x) + nL_n(x) = 0.$$

With the procedure outlined in Section 5 it is easy to establish the recurrence relation

$$nL_n(x) = (2n - 1 - x)L_{n-1}(x) - (n - 1)L_{n-2}(x).$$

But the preceding sections also yield the less commonly known results of Feldheim[5]

$$\sum_{k=0}^{n} (-1)^k C_{n,k} L_k(x) = \frac{x^n}{n!}, \tag{14}$$

and

$$L_n(xy) = \sum_{k=0}^{n} C_{n,k} y^k (1 - y)^{n-k} L_k(x). \tag{15}$$

Formula (14) is attractive symbolically, being

$$\{1 - L(x)\}^n \doteq \frac{x^n}{n!}.$$

It may be used to define the $L_n(x)$, with $L_0(x) = 1$.

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, for which

$$G(x) = x^{-\alpha/2} J_\alpha(2\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!\Gamma(n + \alpha + 1)},$$

may be treated in the same way.

It must be emphasized that our work has no direct bearing on the extremely important orthogonality property of the Laguerre polynomials. Naturally, all properties of the polynomials are, directly or otherwise, results of the generating function employed in defining the polynomials. But, the orthogonality does not follow from the form of the generating function used in (1), as simple counter examples show. Take $G(x) = e^x$, for instance.

8. Some classical formulas. In order to discuss the polynomials which we wish to introduce in the next section, we need certain classical formulas relating to two well-known functions.

Gauss' hypergeometric series or function is

$$F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n)\Gamma(\beta + n)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma + n)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{x^n}{n!}.$$

It satisfies the differential equation

$$[\theta(\theta + \gamma - 1) - x(\theta + \alpha)(\theta + \beta)]F = 0.$$

We shall use the relation

$$\frac{d}{dx} F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; x).$$

A function such as $F(\alpha, \beta+1; \gamma; x)$ in which one and only one of the parame-

ters α, β, and γ, is increased or decreased by unity is said to be contiguous to $F(\alpha, \beta; \gamma; x)$. There are evidently six functions contiguous to $F(\alpha, \beta; \gamma; x)$. For them we shall use a common notation illustrated by

$$F_{\gamma-} = F(\alpha, \beta; \gamma - 1; x),$$
$$F_{\alpha+} = F(\alpha + 1, \beta; \gamma; x).$$

Gauss showed that F and any two of its contiguous functions are related linearly with coefficients at most linear polynomials in x. Of the fifteen relations which Gauss thus obtained the other ten follow from the five below by elimination of $F_{\alpha+}$ between the various pairs. The five relations are:

$$(\alpha - \beta)F = \alpha F_{\alpha+} - \beta F_{\beta+},$$
$$(\alpha - \gamma + 1)F = \alpha F_{\alpha+} - (\gamma - 1)F_{\gamma-},$$
$$[2\alpha - \gamma + (\beta - \alpha)x]F = \alpha(1 - x)F_{\alpha+} + (\alpha - \gamma)F_{\alpha-},$$
$$(\alpha + \beta - \gamma)F = \alpha(1 - x)F_{\alpha+} + (\beta - \gamma)F_{\beta-},$$
$$\gamma[\alpha + (\beta - \gamma)x]F = \alpha\gamma(1 - x)F_{\alpha+} - (\alpha - \gamma)(\beta - \gamma)xF_{\gamma+}.$$

The Pochhammer-Barnes confluent hypergeometric function may be defined by

$$M(\alpha, \gamma, x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{x^n}{n!}.$$

This function satisfies the differential equation

$$[\theta(\theta + \gamma - 1) - x(\theta + \alpha)]\, M = 0.$$

The only additional property of $M(\alpha, \gamma, x)$ which we shall use is Kummer's formula,

$$M(\alpha, \gamma, x) = e^x M(\gamma - \alpha, \gamma, -x).$$

9. A set of hypergeometric polynomials. Let us consider the polynomials $\phi_n(\alpha, \gamma, x)$ defined by

$$e^t M(\alpha, \gamma, xt) = \sum_{n=0}^{\infty} \phi_n(\alpha, \gamma, x) \frac{t^n}{n!}. \tag{16}$$

Whenever the parameters α and γ remain unchanged throughout a relation or discussion we shall omit them and write simply $\phi_n(x)$.

From the earlier sections we have at once the results

$$\phi_n(x) = \phi_n(\alpha, \gamma, x) = \sum_{k=0}^{n} C_{n,k} \frac{(\alpha)_k}{(\gamma)_k} x^k, \tag{17}$$

$$n\phi_{n-1}(x) = (n - \theta)\phi_n(x), \tag{18}$$

$$\theta(\theta + \gamma - 1)\phi_n(x) - nx(\theta + \alpha)\phi_{n-1}(x) = 0, \tag{19}$$

$$[\theta(\theta+\gamma-1)+x(\theta-n)(\theta+\alpha)]\phi_n(x)=0, \tag{20}$$

$$\{1-\phi(x)\}^n \doteq (-1)^n \frac{(\alpha)_n}{(\gamma)_n} x^n, \tag{21}$$

$$\phi_n(xw) \doteq \{1-w+w\phi(x)\}^n. \tag{22}$$

The relations (18) and (19) together yield a pure recurrence relation for $\phi_n(x)$. From (18) in the form

$$\theta\phi_n(x) = n[\phi_n(x)-\phi_{n-1}(x)]$$

it follows that

$$\theta^2\phi_n(x) = n[n\phi_n(x)-(2n-1)\phi_{n-1}(x)+(n-1)\phi_{n-2}(x)].$$

Then direct substitution of these two results into (19) leads to the desired relation,

$$\begin{aligned}(n+\gamma-1)\phi_n(x) = {} & [2n+\gamma-2+(n+\alpha-1)x]\phi_{n-1}(x) \\ & -(n-1)(1+x)\phi_{n-2}(x).\end{aligned} \tag{23}$$

An inspection of equation (20) indicates that $\phi_n(x)$ is a hypergeometric function. In fact, since

$$C_{n,k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = (-1)^k \frac{(-n)(-n+1)\cdots(-n+k-1)}{k!}$$

$$= (-1)^k \frac{(-n)_k}{k!},$$

it is possible to write

$$\phi_n(\alpha,\gamma,x) = \sum_{k=0}^{n} (-1)^k \frac{(-n)_k(\alpha)_k}{(\gamma)_k} \frac{x^k}{k!} = F(\alpha,-n;\gamma;-x). \tag{24}$$

Now it is evident that $\phi_n(x)$ is a terminating hypergeometric series. Indeed, every terminating hypergeometric series is a $\phi_n(x)$, but we must not demand too much of this fact. It is inherent in our definition of $\phi_n(\alpha,\gamma,x)$ that the parameters α and γ be independent of n. Otherwise we would not dare use results obtained on the basis of the generating function and would be forced to reexamine each separate formula. A pertinent case is the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ which can be expressed in terms of $\phi_n(\sigma,\epsilon,x)$ but with σ dependent upon n. Each individual Jacobi polynomial may be written as a ϕ_n but the theory of the ϕ_n's as developed here may not be applied to Jacobi polynomials.

10. Further properties of the polynomials. The following six formulas are immediate results of those properties of $F(\alpha,\beta;\gamma;x)$ given in Section 8. They are obtained by replacing β by $-n$, x by $-x$:

$$\gamma\phi_n'(\alpha, \gamma, x) = n\alpha\phi_{n-1}(\alpha + 1, \gamma + 1, x), \tag{25}$$

$$(\alpha + n)\phi_n(\alpha, \gamma, x) = \alpha\phi_n(\alpha + 1, \gamma, x) + n\phi_{n-1}(\alpha, \gamma, x), \tag{26}$$

$$(\alpha - \gamma + 1)\phi_n(\alpha, \gamma, x) = \alpha\phi_n(\alpha + 1, \gamma, x) - (\gamma - 1)\phi_n(\alpha, \gamma - 1, x), \tag{27}$$

$$\begin{aligned}[2\alpha - \gamma + (\alpha + n)x]\phi_n(\alpha, \gamma, x) &= \alpha(1 + x)\phi_n(\alpha + 1, \gamma, x) \\ &\quad + (\alpha - \gamma)\phi_n(\alpha - 1, \gamma, x),\end{aligned} \tag{28}$$

$$\begin{aligned}(\alpha - n - \gamma)\phi_n(\alpha, \gamma, x) &= \alpha(1 + x)\phi_n(\alpha + 1, \gamma, x) \\ &\quad - (n + \gamma)\phi_{n+1}(\alpha, \gamma, x),\end{aligned} \tag{29}$$

$$\begin{aligned}\gamma[\alpha + (n + \gamma)x]\phi_n(\alpha, \gamma, x) &= \alpha\gamma(1 + x)\phi_n(\alpha + 1, \gamma, x) \\ &\quad - (\alpha - \gamma)(n + \gamma)x\phi_n(\alpha, \gamma + 1, x).\end{aligned} \tag{30}$$

The pure recurrence relation (23) may be obtained by eliminating $\phi_n(\alpha+1, \gamma, x)$ from (26) and (29) and then replacing n by $(n-1)$.

From Kummer's formula for $M(\alpha, \gamma, x)$ we see that

$$e^tM(\alpha, \gamma, xt) = e^te^{xt}M(\gamma - \alpha, \gamma, -xt),$$

and thus that

$$\sum_{n=0}^{\infty}\phi_n(\alpha, \gamma, x)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty}\frac{x^nt^n}{n!}\right)\left(\sum_{n=0}^{\infty}\phi_n(\gamma - \alpha, \gamma, -x)\frac{t^n}{n!}\right).$$

Hence

$$\phi_n(\alpha, \gamma, x) = \sum_{k=0}^{n}C_{n,k}x^{n-k}\phi_k(\gamma - \alpha, \gamma, -x), \tag{31}$$

or

$$\phi_n(\alpha, \gamma, x) \doteq \{x + \phi(\gamma - \alpha, \gamma, -x)\}^n.$$

11. Relation of the $\phi_n(x)$ to some well-known polynomials. The following results are not difficult to verify:

For Laguerre polynomials,

$$L_n(x) = \lim_{\sigma\to\infty}\phi_n(\sigma, 1, -x/\sigma),$$

$$L_n^{(\alpha)}(x) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}\lim_{\sigma\to\infty}\phi_n(\sigma, \alpha + 1, -x/\sigma).$$

For Legendre polynomials,

$$P_n(x) = (x - \sqrt{x^2 - 1})^n\phi_n\left(\frac{1}{2}, 1, \frac{2\sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}}\right).$$

For Hermite polynomials, see Szegö, *op. cit.*, p. 102, and apply the above result for $L_n^{(\alpha)}(x)$.

References

1. L. M. Milne-Thomson, Two classes of generalized polynomials, Proc. Lond. Math. Soc., (2) vol. 35, 1933, 514–522.

2. I am indebted to Professor Brenke for an opportunity to examine his manuscript.

3. Note also that equation (12) may be obtained by equating coefficients of y^n in (13).

4. See G. Szegö, Orthogonal Polynomials, Colloquium Lectures, 1939, 96–98. Our notation agrees with that of Szegö whenever we use classical polynomials.

5. E. Feldheim, Développements en série de polynômes d'Hermite et de Laguerre à l'aide des transformations de Gauss et Hankel, I et II, Nederl. Akad. Wetensch., Proc. 43, 1940, 224–248.

MEAN LENGTHS OF LINE SEGMENTS

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1. Introduction. If we assume a line segment of unit length divided at random into two parts, it is obvious that the mean length of the longer of the two parts will be three-fourths, that of the shorter one-fourth.

If the random division is into three parts, it has been shown* that the mean lengths of the parts, taken in the descending order of magnitude are 11/18, 5/18 and 2/18 respectively.

The general case is more difficult. A line segment is broken at random into $(n+1)$ parts. What are the mean lengths of the separate parts taken in descending order of magnitude?

2. Four segments. We consider first the case when $n=3$. Let us denote the lengths of the four parts taken in descending order of magnitude by x, y, z, w and their mean values by $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{w}$, respectively.

By definition:

$$\bar{x} = \frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz}, \qquad \bar{y} = \frac{\iiint y\,dx\,dy\,dz}{\iiint dx\,dy\,dz}, \qquad \bar{z} = \frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

$$\bar{w} = 1 - \bar{x} - \bar{y} - \bar{z}, \tag{1}$$

where the intervals of integration are subject to the restrictions:

$$x > y, \qquad y > z, \qquad z > w, \qquad w = 1 - x - y - z > 0. \tag{2}$$

Consider now the region of space where coordinates are the values of x, y, z subject to the restrictions (2). The plane $x=y$ separates the region in which $x>y$ from the region in which $x<y$. The plane $y=z$ separates the region in which $y>z$ from the region in which $y<z$, and similar remarks apply to the planes $z=1-x-y-z$ and $1-x-y-z=0$.

The region of integration of the integrals (1) is therefore limited to some one

* See Czuber: Geometrische Wahrscheinlichkeiten und Mittelwerte, p. 207.

of the fifteen regions into which ordinary three-space is divided by the four planes:

$$x = y, \quad y = z, \quad z = 1 - x - z, \quad 1 - x - y - z = 0. \tag{3}$$

Moreover, since x, y, z and $w=1-x-y-z$, are each limited, the region of integration must be the interior of the tetrahedron bounded by the four planes (3).

Let us denote the interior region of the tetrahedron by T. We may now write:

$$\bar{x} = \frac{\int_T x dV}{\int_T dV}, \qquad \bar{y} = \frac{\int_T y dV}{\int_T dV}, \qquad \bar{z} = \frac{\int_T z dV}{\int_T dV}, \tag{4}$$

where $dV = dxdydz$, and T is the region of integration. The direct integration of the integrals in the numerators offer difficulties which may be avoided by observing that the quantities $\bar{x}$, $\bar{y}$, $\bar{z}$ define the centroid of the tetrahedron. They may therefore be computed indirectly from the well-known formulas:

$$\bar{x} = \tfrac{1}{4}\sum x_i, \qquad \bar{y} = \tfrac{1}{4}\sum y_i, \qquad \bar{z} = \tfrac{1}{4}\sum z_i, \qquad i = 1, 2, 3, 4, \tag{5}$$

where x_i, y_i, z_i are the coordinates of the vertices of the tetrahedron. On solving the equations (3) taken three at a time, we obtain the coordinates of the vertices:

$$P_1 = (1, 0, 0), \quad P_2 = (\tfrac{1}{2}, \tfrac{1}{2}, 0), \quad P_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}), \quad P_4 = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}),$$

and thence, by (5)

$$\begin{aligned} &\bar{x} = \tfrac{1}{4}(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) = \tfrac{13}{48}, \qquad \bar{y} = \tfrac{1}{4}(0 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) = \tfrac{13}{48}, \\ &\bar{z} = \tfrac{1}{4}(0 + 0 + \tfrac{1}{3} + \tfrac{1}{4}) = \tfrac{7}{48}, \qquad \bar{w} = 1 - \bar{x} - \bar{y} - \bar{z} = \tfrac{15}{48}. \end{aligned} \tag{6}$$

3. The general case. Let the lengths of

$$x_1, x_2, \cdots, x_n, w,$$

represent the $n+1$ parts, taken in descending order of magnitude, in which a line segment of unit length is divided at random. The mean value of the ith of these parts is:

$$\bar{x}_i = \frac{\int\int \cdots \int x_i dx_1 dx_2 \cdots dx_n}{\int\int \cdots \int dx_1 dx_2 \cdots dx_n}, \qquad i = 1, 2, \cdots, n \tag{7}$$

where the x's are subject to the restrictions

$$x_1 > x_2 > \cdots > x_n > w > 0.$$

Let us interpret the x's as the coordinates of a point in a euclidean space (R_n) of n dimensions. The equation $x_i = x_{i+1}$ may then be interpreted as the equation of a hyper-plane which separates the region of R_n in which $x_i > x_{i+1}$ from the region in which $x_i < x_{i+1}$. The $n+1$ equations

$$x_1 = x_2,\ x_2 = x_3, \cdots, x_n = w = 1 - x_1 - x_2 \cdots - x_n, \tag{8}$$
$$w = 1 - x_1 - x_2 \cdots - x_n = 0$$

being linearly independent, represent the boundary planes of a simplex in R_n, and since each of the x's is limited, x_i being less than unity, the region of integration of (7) must be the interior region of this simplex. The evaluation of the integrals (7) is thus reduced to that of finding the centroid of a simplex in R_n, in terms of the coordinates of the vertices of the simplex. These may be readily computed by solving the equations (8) in sets of n.* Let P_i denote the vertex of the simplex determined by those equations, omitting the ith. An easy computation gives the coordinates of P_i as tabulated below.

	x_1	x_2	x_3		x_{n-2}	x_{n-1}	x_n
P_1	1	0	0	$\cdots$	0	0	0
P_2	$\frac{1}{2}$	$\frac{1}{2}$	0	$\cdots$	0	0	0
P_3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\cdots$	0	0	0
. . . . .							
P_{n-1}	$\frac{1}{n-1}$	$\frac{1}{n-1}$	$\frac{1}{n-1}$	$\cdots$	$\frac{1}{n-1}$	$\frac{1}{n-1}$	0
P_n	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\cdots$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$
P_{n+1}	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\cdots$	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$

Applying now the theorem referred to we obtain the mean values sought.

$$\bar{x}_1 = \frac{1}{n+1}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{n+1}\right)$$
$$\bar{x}_2 = \frac{1}{n+1}\left(0 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{n+1}\right)$$
$$\bar{x}_3 = \frac{1}{n+1}\left(0 + 0 + \tfrac{1}{3} + \cdots + \frac{1}{n+1}\right)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\bar{x}_n = \frac{1}{n+1}\left(0 + 0 + 0 + \cdots + \frac{1}{n} + \frac{1}{n+1}\right)$$
$$\bar{w} = 1 - \bar{x}_1 - \bar{x}_2 - \cdots - \bar{x}_n = \frac{1}{(n+1)^2}.$$

* The assumption is made that the centroid formulas are valid for n-dimensional space.

AN ANALYTIC GEOMETRY FOR N VARIABLES

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While it is known* that we may set up a one-to-one correspondence between the points in n-space and those on a line or plane, no device for doing this of which the writer is aware would yield anything practical in the way of a picture. Yet any system which provides a framework for a plane and solid analytic geometry involving n variables should broaden the base for the impact of euclidean geometry upon algebra, and vice versa. With this general goal in mind the writer has experimented with various coordinate systems, seeking one which meets satisfactorily the tests of simplicity, consistence with the developed subject for the cases of two and three variables, and practical usefulness in the interpretation of such things as partial derivatives. Herewith is presented his best result in the light of these criteria.

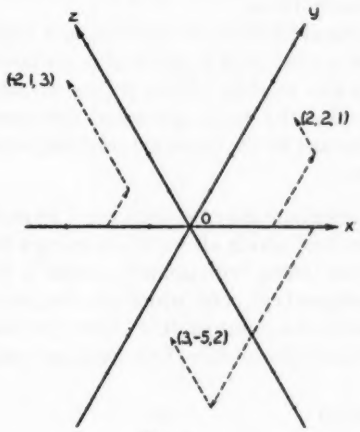

FIG. 1

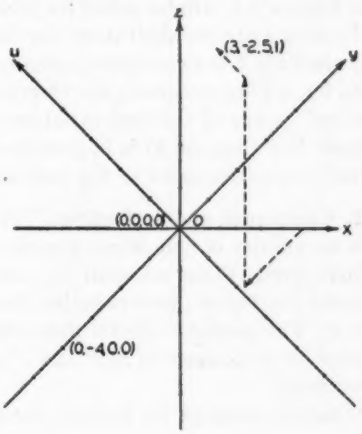

FIG. 2

A. THE TWO-DIMENSIONAL CASE

1. The n-axes plane. The basic feature of the plan is a set of n equally-spaced axes passing through the origin on *the n-axes plane*, with one axis horizontal, and with the positive direction always upward along a slant or vertical axis and to the right on the horizontal one. Figures 1 and 2 show the planes with three and four axes respectively. Some sample points are shown to indicate the method of locating them by means of the assigned coordinates, which specify the positive or negative distances to be measured parallel to the respective axes. Except in the case of two variables, here always denoted by X and Y, the coordinates x, y, z, u, v and w are used in order as needed to correspond with successive axes read counterclockwise, beginning with the horizontal one. In the

* Ch.-J. de la Vallée Poussin, Cours d'Analyse Infinitésimale, Paris, 1914, p. 49.

general case, they will be designated as x_i $(i=1, 2, \cdots, n)$. The angle between successive axes is always π/n.

A given set of coordinates x_i, as applied in vector sequence from the origin, will of course, as in the case of vectors, lead always to the same point, regardless of the order of application. Hence each set of coordinates designates one and only one point, though the converse is not true except when $n=2$. On the n-axes plane the lack of this converse feature is responsible for a "piling of points on top of each other" in the graphs of equations in three or more variables. We shall see, however, that this apparent flaw may be practically nullified by use of a simple device.

The usual definition of a locus must now be modified slightly, thus:

The locus of an equation in n variables is the totality of points on the n-axis plane which have coordinates satisfying the equation.

Such coordinates of a locus point, as distinguished from the others it will have when $n>2$, will be called its *proper* coordinates.

To anticipate, we shall show that the normal locus of an equation in n variables, when $n>2$, is a continuous area such as a *filled circle* (a circle plus the points inside it), a *filled parabola, etc.* Degenerate loci include all the points, straight lines and curves of the two-variables case. But the appearance and derivation of these loci (Section **3**) is in practice irrelevant to the interpretation of partial derivatives, as discussed in the next section.

2. Complanes and derivatives. The n-axes plane may be considered as made up of an infinity of coincident planes, on each of which all variables except two are held fixed. These we shall call *complanes* (from "component planes"). For example, the 3-axes plane contains the complane $(xy)_{z=2}$ on which are the points $(x, y, 2)$. The special origin for this complane is the point $(0, 0, 2)$. Only two axes are used for it, designated as x' and y' and drawn parallel to the x-axis and y-axis respectively.

Whatever may be the locus of the equation

$$f(x, y, z) = 0, \tag{1}$$

on the 3-axes plane, only the curve

$$f(x, y, 2) = 0 \tag{2}$$

appears on the complane $(xy)_{z=2}$. And when all the complanes $(xy)_{z=k}$ $(-\infty<k<\infty)$ are placed on each other like so many sheets of glazed paper of zero thickness, the various curves taken together black out the total locus. It is natural to speak of these curves as *traces* of the locus on the respective complanes. In addition to those mentioned, two more groups of traces would appear in the complanes $(xz)_{y=k}$ and $(yz)_{x=k}$; but neither of these would add any points to the total locus as described above, since every real set (x, y, z) satisfying (1) fixes a locus point on a curve in one or more of the complanes $(xy)_{z=k}$.

The geometric interpretation of a partial derivative follows from an extension of the 2-axes practice. To state it precisely some definitions are needed.

Let x and y stand for any two of the variables x_i, and let (xy) represent any complane using these variables. Designate by m_{xy} the "slope" of a straight line L in (xy), with the first letter in the subscript standing for the independent variable. Then

$$m_{xy} = (y_2 - y_1)/(x_2 - x_1)$$

where (x_1, y_1) and (x_2, y_2) are any two distinct points on L.

It will now be evident that $\partial y/\partial x$ at a point P having specified coordinates is simply the slope m_{xy} of the line tangent at P to the curve in the complane (xy) containing P. Similarly, $\partial x/\partial y = m_{yx}$ for the same line.

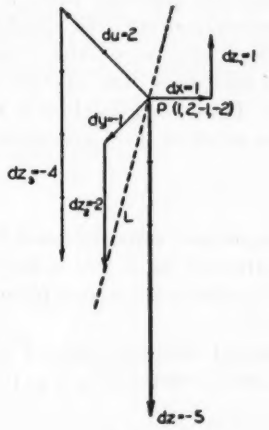

FIG. 3

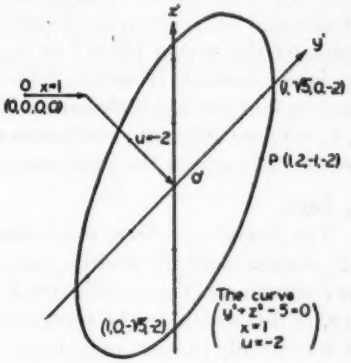

FIG. 4

To illustrate, Figure 3 shows visual interpretations of some of the quantities: $\partial z/\partial x$, $\partial z/\partial y$, $dx = 1$, $dy = -1$, $du = 2$, and the total differential dz, at the point $P(1, 2, -1, -2)$ on the locus of

$$f(x, y, z, u) \equiv x^2 + y^2 + z^2 + u^2 - 10 = 0. \tag{3}$$

The value of $\partial z/\partial y$, for example, is the slope m_{yz} at P of the trace of (3) which appears in the complane $(yz)_{x=1,\ u=-2}$, and whose equation is

$$f(1, y, z, -2) \equiv y^2 + z^2 - 5 = 0. \tag{4}$$

Since $\partial z/\partial y = 2$ at P, this is the slope m_{yz} of the dotted line L. Similarly, at P, $\partial z/\partial x = 1$ and $\partial z/\partial u = -2$. Now let $dz_1 = (\partial z/\partial x)dx$, $dz_2 = (\partial z/\partial y)dy$, and $dz_3 = (dz/\partial u)du$, so that

$$dz = dz_1 + dz_2 + dz_3. \tag{5}$$

For the assigned value of dx, $dz_1 = 1$, and hence the directed segment representing

dz_1 must be drawn in the positive z-direction. The segments for $dz_2 = -2$, $dz_3 = -4$, and $dz = -5$ are of course oppositely directed.

The locus of (4) is shown in Figure 4. This curve is an ellipse whose equation in X and Y can be found by use of (4) and (7). But this method involves cumbersome algebra and is unnecessary for a rough sketch, since it is simpler to plot directly on the complane some of the points $(1, y, z, -2)$ whose coordinates satisfy (4).

The traces of (3) in the complanes $(xz)_{y=2,u=-2}$, $(yz)_{x=1,u=-2}$, and $(uz)_{x=1,y=2}$ are ellipses (one of them a circle) intersecting at P. Each of them, being a part of the total locus of (3), must lie within that locus. The latter, incidentally, is shown in Section 3 to be a filled circle, of radius $\sqrt{20}$, with center at the origin.

Note: By a given set of coordinates (an algebraic concept) there is indicated one and only one point-location (a geometric concept). In speaking of "a point on the n-axes plane," as in our definition of "locus," the geometric aspect is emphasized. However, when a point is said to be "on" or "not on" a given complane the algebraic aspect is dominant. Thus $(0, 0, 0)$, $(1, -1, 1)$ and $(-1, 1, -1)$ are different coordinates of the same point on the 3-axes plane, but no two of them are on the same complane.

3. Loci.

a. *The general case.* Over every n-axes system we may superimpose a 2-axes, or XY, system, with the positive and negative halves of the X and x_1 axes coinciding respectively. The coordinates X and Y of a point on the n-axes plane refer always, in this paper, to the superimposed system.

It may readily be seen, upon inspection of Figure 1, that for a point $P(x, y, z)$ on the 3-axes plane the relation between the two sets of coordinates is as follows:

$$\begin{aligned} X &= x\cos 0^\circ + y\cos 60^\circ + z\cos 120^\circ; \\ Y &= x\sin 0^\circ + y\sin 60^\circ + z\sin 120^\circ. \end{aligned} \tag{6}$$

Similarly, for the n-axes case,

$$X = \sum_{i=1}^{n} x_i \cos\theta_i; \qquad Y = \sum_{i=1}^{n} x_i \sin\theta_i; \qquad \theta_i = (i-1)\pi/n, \tag{7}$$

where the θ_i are in order the least positive angles made with the positive half of the x_1 axis by the successive axes.

We may call (7) *the fundamental equations of the n-axes plane.* Of course, for the 2-axes system they yield the trivial equations: $X = x_1$; $Y = x_2$.

It may be noted that (7) would still be valid with other choices for the θ_i. This might indeed be kept in mind as a factor of flexibility yielding systems adapted to special problems. However, some advantages of the equal spacing of axes will appear in what follows, as in Theorem 1.

Here, as well as in the solid geometry variant introduced in Part B, something new is added to the technical tools of orthodox analytic geometry. For we need but substitute for X and Y, in any two-variable equation, the values given in (7), and we get an n-variable equation *with the same locus*, which will now, for

$n>2$, be a degenerate form of a typical locus. This interesting result could conceivably have direct physical or mechanical applications.

b. *Quadric loci on the 3-axes plane.* From (6) we get, for the 3-axes case,

$$2x + y - z = 2X; \qquad y + z = 2Y/\sqrt{3}. \tag{8}$$

We seek now the total locus of

$$f(x, y, z) = 0. \tag{9}$$

When f is a second degree function of the independent variables, whatever their number, the locus will be called a *quadric locus*. Since such loci often fail to cover the whole plane, the problem posed is to find the equation of the bounding curve in terms of X and Y. This may be done for (9) by eliminating two of the variables x, y and z from (8) and (9) and equating to zero the discriminant of the quadratic in the remaining variable. The device is illustrated as applied to

$$f(x, y, z) \equiv x^2 + y^2 + z^2 - a^2 = 0. \tag{10}$$

From (8) we get

$$y = Y/\sqrt{3} + X - x; \qquad z = Y/\sqrt{3} - X + x. \tag{11}$$

These values, substituted in (10), yield

$$Ax^2 + Bx + C = 0, \tag{12}$$

where

$$A = 3; \qquad B = -4X; \qquad C = 2X^2 + 2Y^2/3 - a^2. \tag{13}$$

Evidently x, as well as, through (11), y and z, will be real for any real pair X and Y so chosen that $D \equiv B^2 - 4AC \geqq 0$. Here $D = 12a^2 - 8X^2 - 8Y^2$. For $D = 0$,

$$X^2 + Y^2 = 3a^2/2. \tag{14}$$

Thus any point (X, Y) on the circle (14) will yield one and only one set of coordinates (x, y, z) satisfying (10). Any point inside the circle will have two such sets of proper coordinates. For example, the proper coordinates of the point (0, 0) as solutions of (10) are $(\pm a/\sqrt{3}, \mp a/\sqrt{3}, \pm a/\sqrt{3})$, and not (0, 0, 0).

As a second example of a typical result, we find that the locus of the equation $x = y^2 + z^2$ is the filled parabola

$$Y^2 = \tfrac{3}{2}(X + \tfrac{1}{6}). \tag{15}$$

c. *A quadric locus on the 4-axis plane, and its generalization.*

The algebraic solution for the total locus of an equation in n variables is in general more difficult when $n>3$. Some available methods may be suggested by the following treatment of the equation

$$x^2 + y^2 + z^2 + u^2 = a^2. \tag{16}$$

For $n=4$, equations (7) yield

$$\sqrt{2}\,x + y - u = \sqrt{2}\,X; \qquad y + \sqrt{2}\,z + u = \sqrt{2}\,Y. \tag{17}$$

Eliminating y and z from (16) by means of (17), and equating to zero the discriminant of the quadratic in u, we have

$$8(-2X^2 - Y^2 + 4Xx - 4x^2 + 2a^2) = 0. \tag{18}$$

This is equivalent to the family of ellipses

$$2(X - x)^2 + Y^2 = 2(a^2 - x^2) \tag{19}$$

representing the bounding curves in terms of the parameter x. Evidently $-a \leqq x \leqq a$, as is also apparent by inspection of (16). For any permissible x the locus area is the corresponding filled ellipse, and the total locus is the filled envelope of (19).

Seeking the maximum (and minimum) X for a given Y, we consider X in (19) as a function of x, and set $dX/dx = 0$, whence $x = X/2$. Replacing x by $X/2$ in (19), we have

$$X^2 + Y^2 = 2a^2. \tag{20}$$

Thus the locus of (16) is the filled circle (20).

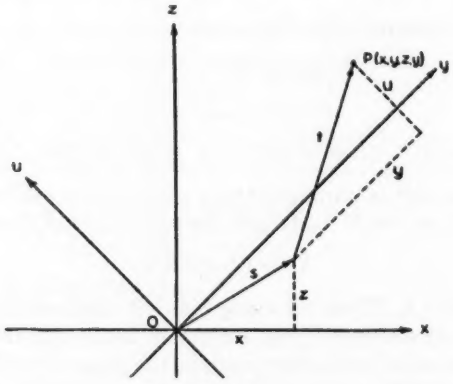

FIG. 5

The important role which geometry may play in this field is well illustrated by a geometric proof of result (20), which leads to the following generalization:

THEOREM 1. *When n is even and exceeds 2, the locus of*

$$\sum_{i=1}^{n} x_i^2 = a^2 \tag{21}$$

is the filled circle

$$X^2 + Y^2 = na^2/2. \tag{22}$$

The proof for the case $n=4$ follows: Consider a point $P(x, y, z, u)$ (Figure 5) on the locus of (16). Since the pairs x, z and y, u are mutually perpendicular

$$a^2 = (x^2 + z^2) + (y^2 + u^2) = s^2 + t^2. \tag{23}$$

For any given pair s, t satisfying (23), various choices of the pairs x, z and y, u are available. The ones which place P as far as possible from 0 are those which make one straight segment from the given segments s and t, so that $OP=s+t$, or, by (23),

$$OP = s + \sqrt{a^2 - s^2}. \tag{24}$$

By calculus, OP reaches its maximum when $s=t=a/\sqrt{2}$, whence $OP=\sqrt{2}a$.

In the general case, with n even, the pairs $x_i, x_{n/2+i}$ $(i=1, 2, \cdots, n/2)$ are mutually perpendicular, and hence, by grouping the variables properly in pairs,

$$a^2 = (x_1^2 + x_{n/2+1}^2) + \cdots + (x_{n/2}^2 + x_n^2) = s_1^2 + s_2^2 + \cdots + s_{n/2}^2. \tag{25}$$

For a given choice of the s_i satisfying (25), OP will reach its maximum when

$$OP = s_1 + s_2 + \cdots + s_{n/2}. \tag{26}$$

The rest of the proof is by induction. Given $k_m^2 = s_1^2 + s_2^2 + \cdots + s_m^2$, where the k_i $(i=1, 2, \cdots, m)$ are constants, assume that, for each value of m,

$$OP = s_1 + s_2 + \cdots + s_m \tag{27}$$

attains its maximum value, which is $\sqrt{m}\, k_m$, when $s_1=s_2=\cdots=s_m$. The assumption is verifiable by calculus when $m=2$. Let $k_m=t$ and $k_{m+1}=a$. Then

$$a^2 = (s_1^2 + \cdots + s_m^2) + s_{m+1}^2 = t^2 + s_{m+1}^2 = t^2 + (a^2 - t^2), \tag{28}$$

and

$$OP = (s_1 + \cdots + s_m) + s_{m+1} = \sqrt{m}\, t + \sqrt{a^2 - t^2}. \tag{29}$$

Treating OP as a function of t, we find that it reaches its maximum when $t=a\sqrt{m/(m+1)}$, so that $OP=a\sqrt{m+1}$ and $s_{m+1}=\sqrt{a^2-x^2}=a/\sqrt{m+1}$ $=OP/(m+1)$, completing the induction.

Note: It seems likely that Theorem 1 holds also when n is odd. By way of partial verification, the writer has proved that the locus of (21) contains the circle (22) when $n \geqq 2$.

d. *Loci of first degree equations.*

THEOREM 2. *The locus of the equation*

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n = A_0 \tag{30}$$

is normally the whole n-axes plane. It is a straight line if and only if the rank of the matrix

$$\left\| \begin{matrix} A_1 & \cdots & A_n \\ \cos\theta_1 & \cdots & \cos\theta_n \\ \sin\theta_1 & \cdots & \sin\theta_n \end{matrix} \right\| \tag{31}$$

is two.

Proof. The elements in the rows of (31) are the coefficients of the homogeneous terms in x_i $(i=1, 2, \cdots, n)$ in equations (30) and the two parts of (7). If the rank is 3, all of the x_i except three which are properly selected, say x, y and z, may be set equal to zero. The equations may then be solved, giving x, y and z as linear functions of X and Y. Thus all the points of the plane are on the locus, since values for X and Y may be assigned at will. If, on the other hand, the rank is 2 (it cannot be 1 in view of the linear independence of the two bottom rows), there exist constants A, B and C, not all zero, such that $AA_i+B\cos\theta_i+C\sin\theta_i=0$, and hence

$$A\sum_{i=1}^{n} A_i x_i + B\sum_{i=1}^{n} x_i\cos\theta_i + C\sum_{i=1}^{n} x_i\sin\theta_i = 0, \tag{32}$$

since the total coefficient of each x_i is zero. Therefore unless, with reference to the terms A_0, X and Y in (30) and (7),

$$AA_0 + BX + CY = 0, \tag{33}$$

the three equations in (30) and (7) are inconsistent. In other words, only points of (33) are on the locus of (30).

An example of such a degenerate case is the line

$$2x + (2\sqrt{3}+1)y + (2\sqrt{3}-1)z = 5, \tag{34}$$

whose equation is two variables is

$$2X + 4Y = 5. \tag{35}$$

Here (35) was written first and then changed to (34) by use of (8).

4. Simultaneous equations. A point is on the *intersection* of two simultaneous equations if it has a set of coordinates which satisfy both equations.

As in solid analytic geometry, the intersection of the loci of two equations in three variables will normally be a curve; but in this case the curve lies directly on the 3-axes plane instead of twisting inconveniently into space.

Consider, for example, the planes

$$5x + 2y - z = 2; \qquad x + 2y + z = 2. \tag{36}$$

Eliminating z and then y, we get successively: $y=1-3x/2$ and $z=2x$. The three general coordinates of an intersection point are therefore x, $1-3x/2$, and $2x$; so that by assigning two different values to x we find at once two points on the line of intersection. The proof that this intersection is the straight line

$$X + \sqrt{3}Y = 2 \tag{37}$$

may be carried through as indicated in the next paragraph.

Evidently a straight line or curve will be the normal type of intersection of the loci of $n-1$ equations in n variables, where $n \geqq 3$. For one letter may be chosen as a parameter in terms of which alone each of the others may be expressed after the proper eliminations. Then by means of the form of (7) which applies, X and Y may be expressed in terms of the chosen letter, thus yielding the equation of the intersection in parametric form. When the parameter is eliminated, if this is convenient or desirable, the equation of the intersection appears directly in terms of X and Y.

If the number of variables exceeds the number of simultaneous equations by more than one, the intersection will normally be an area. Numerous special cases of course occur, especially when the loci of some of the simultaneous equations are degenerate.

B. THE THREE-DIMENSIONAL VARIANT

In this system we shall use, in addition to the equally-spaced axes on a plane, one extra axis which is perpendicular to that plane at the origin. The use of the capital Z to designate this special axis serves not only to show its unique status when the number of variables exceeds three, but also to give a consistent notation (with variables X, Y and Z) for the 3-axes case, which is that of orthodox solid analytic geometry.

It is true that analogy suggests for the solid case an equal spacial distribution of the axes. But the scheme here suggested, besides being more practical algebraically, permits the use of all the theory developed for the plane case. In addition, it exhibits in a useful way the special status of one variable, to be represented by Z, which is expressed explicitly as a function of the others.

The locus of an equation in four or more variables is normally the totality of points in a solid which may or may not be bounded. The revealing cross-sections of such loci made by the planes $Z=k$ are often sufficient to indicate their general contours, and may even suggest the equations of the bounding surfaces. For example, consider the equation

$$Z = x^2 + y^2 + z^2 + u^2 - a^2. \tag{38}$$

By Theorem 1, we know that the trace of this locus on the plane $Z=k$ is the filled circle

$$X^2 + Y^2 = 2(a^2 + k). \tag{39}$$

It follows that the locus of (38) is the filled paraboloid

$$Z = (X^2 + Y^2)/2 - a^2, \tag{40}$$

since the trace of (40) in the plane $Z=k$ is identical with (39).

It is evident that in this system the interpretation of a partial derivative not involving Z is exactly that of Section A 2, as applied in the proper plane $Z=k$. To interpret a partial derivative of Z we use a Z-plane, or a plane parallel to the Z-axis. For example, in the solid 5-axes system, the Z-plane $Zx)_{y=4,\ z=0,\ u=4}$ has a

2-axes system whose conveniently perpendicular x' and z' axes are parallel respectively to the x and Z axes, and whose origin is the point (0, 4, 0, 4, 0), the Z-coordinate being always last. The trace of the locus of

$$f(x, y, z, u, Z) = 0 \tag{41}$$

on this Z-plane is the curve

$$f(x, 4, 0, 4, Z) = 0 \tag{42}$$

whose slope at $x = x_1$ and $Z = Z_1$ is the value of $\partial Z/\partial x$ at the point $(x_1, 4, 0, 4, Z_1)$.

It should be noted that many different Z-planes may coincide, with each one contributing its special curve to the total locus. Thus the Z-planes are somewhat analogous to the complanes on the planes $Z = k$. For example, the Z-planes $Zx)_{y=4,z=0,u=4}$ and $Zx)_{y=0,z=4\sqrt{2},u=0}$ coincide; but the former contains the curve (42) and the latter the curve

$$f(x, 0, 4\sqrt{2}, 0, Z) = 0. \tag{43}$$

Again, to illustrate graphically the relation between any three variables which include Z, we may use a 3-axes *subsystem*, in which all variables except Z and two others are held fixed. For example, the subsystem $yzZ_{x=4,u=3}$ has its origin at the point (4, 0, 0, 3, 0), with its axes y', z' and Z' parallel, respectively, to the y, z and Z axes. The portion of the locus of (41) which appears as a trace in this subsystem is the surface

$$f(4, y, z, 3, Z) = 0. \tag{44}$$

As on the n-axes plane, equations in n variables having degenerate loci may be obtained readily. In this case, substitutions (7) are made upon equations in X, Y and Z, with Z remaining unchanged.

To summarize, in this paper we have sketched the essential features of a plane and solid analytic geometry for n variables. Sample loci have been obtained, sample methods have been suggested, and some general results have been stated as theorems.

It is clear that the investigation of the two-dimensional phase should come first, since knowledge of the loci on the n-axes plane greatly facilitates the study of the loci in space.

Question and answer. How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?—Einstein.

Nature has not embarrassed itself with mathematical difficulties.—Fresnel.

Man and Beast. So celebrated was this proposition [that the square root of 2 is irrational] among the ancient philosophers that Plato declared anyone ignorant of it was not a man but a beast.—Isaac Barrow.

Irrational numbers do not exist.—Leopold Kronecker. *Contributed.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE "TIME-UNIT" SYSTEM IN NAVIGATION

R. A. ROSENBAUM, U.S.N.R.

1. Introduction. In celestial navigation it is rare that the observations for a fix are made simultaneously, so that the lines of position resulting from the earlier observations must be "advanced" to the time of the last. This is often done as follows (Fig. 1): Any point G is chosen on l, the line of position. In the direction of the ship's track, GG' is measured equal to the distance covered during the time-interval for which l is to be advanced. Then l', the line parallel to l through G', is the required advanced line, for, if P is the actual position of the ship on l, its advanced position, P', will lie on l'.

This method has the disadvantage, in aerial navigation, that frequently neither the track nor the speed which the aircraft is making good is accurately known. The error resulting from use of an incorrect track and speed will ordinarily have a negligible effect on the usefulness of the fix, but some perfectionists like to use the following method, when it is applicable.

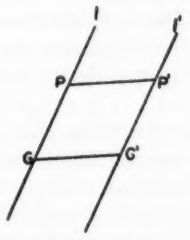

FIG. 1

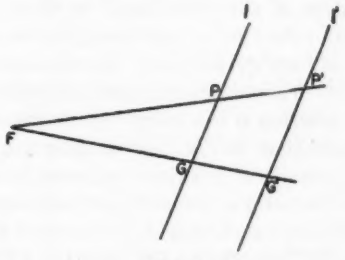

FIG. 2

2. The "Time-Unit" System. This method may only be used when it can be assumed that track and speed have been constant since the last fix. Suppose, in Fig. 2, that F is the position of a fix at time T, and that l is a line of position obtained t minutes later. This means that the aircraft is at F at time T and is somewhere on l at $T+t$. Suppose further that it is desired to advance l for τ minutes, *i.e.*, to find the possible positions for the aircraft at time $T+t+\tau$. The method consists in choosing any convenient unit of length on a marked straight-edge to represent one minute of time, and to move the straight-edge about on the chart until a point G is obtained on l such that $FG=t$, according to

the chosen unit. Then FG is prolonged to G' such that $GG'=\tau$, and l' is drawn parallel to l through G'. The advanced line of position is l', for:

Suppose that the actual track is FP, that the observer t minutes after the fix is at P, and that $t+\tau$ minutes after the fix he is at P'. Then, under the assumption of constant track and speed, the distance covered is proportional to the elapsed time; hence,

$$FG/GG' = t/\tau = FP/PP'.$$

Therefore, P' lies on l'.

Note that this method requires that the track and speed of the aircraft be constant, but it does not require that the navigator know their values.

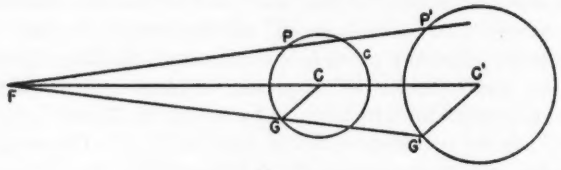

FIG. 3

3. Extension. The line of position resulting from the usual celestial observation is an approximation to part of a large circle of position. In case the altitude of the observed body is close to 90°, the circle is small, and a straight-line approximation is not sufficiently accurate—the circle itself must be plotted. It is the purpose of this paragraph to show how the "time-unit" method may be extended to the case of advancing circles of position. The method illustrates the perhaps unfamiliar fact that the advanced line or curve of position need not be simply the original one displaced; in this case it will appear that the advanced circle of position is not congruent to the original.

Suppose that, in Fig. 3, F is again the position of a fix at time T; c, with center C, is a circle of position t minutes later; and it is desired to advance c for τ minutes. As before, choose a unit of length to represent one minute; with this scale find G on c such that $FG=t$; extend FG to G' such that $GG'=\tau$; and through G' draw $G'C'$ parallel to GC, meeting FC produced at C'. Then C' is the center and $C'G'$ the radius of the advanced circle. The reasoning is based on proportions as before.

Note by the Editor. While his note *Checking the SAS Case in Trigonometry* in the April issue of this MONTHLY was in press, Professor Eves called my attention to the fact that Professor E. J. Moulton had pointed out the same discrepancy in the customary check of the *SAS* case in trigonometry in this MONTHLY, vol. 31, 1924, p. 292. M.J.W.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1943–44

Mathematical Society, Hofstra College

During the year the following talks were given:

Mathematics in art, an illustrated lecture by Professor Constant Van de Wall of the Fine Arts department

Greek mathematics, by Harry Durham

Mathematical logic, by Robert Ackerson

Signifying nothing, by Professor L. F. Ollmann.

One meeting was devoted to a discussion of mathematical brain-teasers. A joint picnic was held with *Kappa Mu Epsilon*.

Officers for the year were: President, Edith Hufman; Vice-President, Joseph Consoli; Secretary, Dorothy McFarland; Treasurer, Vera Pabo; Historian, Rheba Handler; Faculty Adviser, Professor E. R. Stabler.

Kappa Mu Epsilon, Hofstra College

The following papers were presented during the year to the New York Alpha Chapter of *Kappa Mu Epsilon*:

Mechanics of structural design, by Professor A. D. Capuro

Postulational methods, by Professor E. R. Stabler

Mathematics of finance, by Professor L. F. Ollmann

Photography, by Mr. Stanley Rodgers of the Physics department

The motion of projectiles, by Professor W. L. Ayres of Purdue University.

In addition, a banquet was held, and there was a joint picnic with the Mathematical Society.

Officers were as follows: President, Edith Hufman; Vice-President, Harry Durham; Secretary, Wanda Scala; Treasurer and Historian, Jean Ruppel; Corresponding Secretary, Professor E. R. Stabler; Faculty Sponsor, Professor A. D. Capuro.

Pi Mu Epsilon, Washington Square College, New York University

The Executive Council agreed that the war-time curtailment of Chapter activities should be continued during this year, omitting the former annual banquet and inter-scholastic contest. Chapter meetings were restricted to two lectures, two business sessions, and two induction ceremonies at which fifteen members were inducted. The two lectures, with refreshments following, were:

Demonstrations of soap film in connection with minimal surfaces and problems of stability, by Professor Richard Courant.

The foundations of probability, by Professor Hans Reichenbach of the University of California, at a meeting held jointly with the New York University Philosophical Society.

Officers elected for 1944–45 were: Chapter Director, Professor W. M. Maiden: Vice-Director, Naomi Rosenstein; Treasurer, Peter Lax; Secretary, Eleanor Karasak; Permanent Secretary, Professor F. W. John.

Pi Mu Epsilon, University of Arkansas

Because of the fewness of older members in our University of Arkansas chapter and the heavy schedules imposed by the Air Corps on our faculty members there has been no presentation of papers this past year. The annual banquet and initation of new members was revived this year and took place on April 30. Eight candidates were initiated and each presented a short humorous theme on some phase of mathematics which served as the program for the banquet.

Actually there were fifteen candidates initiated this year, but the other group was initiated at a special meeting. The necessity of the other initiation arose from the quarter system here at the University. It is interesting to note that a greater percentage of girls were initiated this year than ever before.

Officers for the year were: Director, J. W. Keller; Vice-Director, Dan Welchel; Secretary, Roger Harris; Treasurer, Carl Gamel.

AN APPROACH TO THE NORMAL CURVE AND THE CYCLOID

J. S. FRAME, Michigan State College

1. Attractive forces. At first sight the parabola, the cosine curve, the cycloid, and the areas and ordinates of the normal curve of error do not seem to have much in common, except that they all seem to be popular subjects for exercises in a first course in the calculus. But the inner harmony of mathematics is illustrated once again in the fact that these curves may all be considered as different special cases of the parametric relationship between the distance y and the time t for a particle moving in a straight line under an attractive force proportional to a power of the distance.

At a combined meeting of the Mathematics Club and the Physics Club, four speakers might each present one of these four special curves from the common viewpoint, and a fifth speaker might discuss the general case which is treated below.

The common ancestor of these curves is the differential equation

$$\frac{dv}{dt} = -ky^{n-1}, \quad \text{where} \quad v = \frac{dy}{dt}. \tag{1}$$

It will be convenient to take $t=0$ when $v=0$, and let $y=a$ at that time. Since the velocity v will be negative for small positive t, we shall introduce a positive parameter u defined by the equations

$$u = -v/c, \qquad c^2 = ka^n. \tag{2}$$

Multiplying both sides of equation (1) by dy/c^2, dt may be eliminated and we obtain

$$\frac{vdv}{c^2} = -\frac{y^{n-1}dy}{a^n}. \tag{3}$$

Integrating, and noting that $y=a$ when $v=0$, we have

$$\frac{v^2}{2c^2} = \frac{u^2}{2} = \frac{1}{n}\left\{1 - \left(\frac{y}{a}\right)^n\right\}, \qquad n \neq 0, \tag{4}$$

or

$$\frac{v^2}{2c^2} = \frac{u^2}{2} = \ln\frac{a}{y}, \qquad n = 0. \tag{4'}$$

2. The parabola and the cosine curve. For $n=1$, equation (1) is simply the law for the motion of a body falling vertically from a height a under a constant gravitational force. The path of the motion is a straight line, but the graph expressing y as a function of t is a *parabola*.

For $n=2$, equation (1) represents the simple harmonic motion of a weight hanging on a stretched spring and oscillating up and down with amplitude a about its position of equilibrium. The graph expressing y in terms of t is a *cosine curve*. These facts are well known to teachers of the calculus, and should also be known by students who have had a year of calculus.

3. The cycloid and the normal curve. It is not common knowledge, however, nor is it mentioned in most calculus texts, that the cases $n=-1$ and $n=0$ of equation (1) lead respectively to the *cycloid* and the *normal curve of error*. The case $n=-1$ is the *inverse square law* which governs the motion of a body falling to the earth from a distance, or the motion of a charged particle attracted to a particle of opposite charge. The case $n=0$ is the *inverse first power law* of force, and can be realized physically in the motion of a charged particle attracted to a long wire of opposite charge.

Perhaps the natural way to integrate equations (4) and (4') would be to express $dt=dy/v$ in terms of y and integrate. But this procedure leads to square roots under the integral sign. Thus

$$t = \frac{-1}{c}\int_a^v \frac{dy}{\sqrt{2(a-y)/y}} \quad \text{for} \quad n = -1, \tag{5}$$

and

$$t = \frac{-1}{c}\int_a^v \frac{dy}{\sqrt{2\ln(a/y)}} \quad \text{for} \quad n = 0. \tag{5'}$$

Simpler results are obtained by expressing y and t both parametrically in terms of u. Thus for $n=-1$, we have

$$(6)\qquad \frac{u^2}{2}=\frac{a}{y}-1,\quad y=\frac{a}{1+u^2/2},\quad t=\int_0^v\frac{dy}{-cu}=\frac{a}{c}\int_0^u\frac{du}{(1+u^2/2)^2}.$$

The substitution $u=\sqrt{2}\tan\phi$ gives

$$y=a\cos^2\phi,\qquad t=\frac{a}{c}\sqrt{2}\int_0^\phi\cos^2\phi d\phi=\frac{a\sqrt{2}}{4c}\int_0^\phi(1+\cos 2\phi)d2\phi,$$

or

$$(7)\qquad \sqrt{2}\,ct=\tfrac{1}{2}a(2\phi+\sin 2\phi),\qquad y=\tfrac{1}{2}a(1+\cos 2\phi).$$

With y as ordinate and $\sqrt{2}ct$ as abscissa, these are parametric equations of a cycloid which has the top point of its arch on the y-axis, and for which 2ϕ is the angle of rotation of a rolling wheel which generates the curve.

Similarly for $n=0$, we have from (4′) and (5′) the parametric equations

$$(6')\qquad \frac{u^2}{2}=\ln\frac{a}{y},\qquad y=ae^{-u^2/2},\qquad t=\int_0^v\frac{dy}{-cu}=\frac{a}{c}\int_0^u e^{-u^2/2}du,$$

or

$$(7')\qquad \frac{ct}{a}=\int_0^u e^{-u^2/2}du,\qquad \frac{y}{a}=e^{-u^2/2}.$$

We see that y/a and ct/a are respectively the ordinate and area for the unit normal curve of error, in which $u=-v/c$ is the abscissa. Since these functions are tabulated in parallel columns in many mathematical tables, the numerical values for plotting y in terms of t are readily accessible.

4. The solution of the general case. In general we may express both y and t in terms of the parameter $u=-v/c$, and we obtain from equation (4)

$$(8)\qquad y=a(1-nu^2/2)^{1/n},\qquad ct=a\int_0^u(1-nu^2/2)^{(1-n)/n}du.$$

In particular, for $n=1$, equation (8) defines the parabola

$$(8')\qquad y=a(1-u^2/2),\qquad ct=au;\quad \text{or,}\quad y=a-c^2t^2/2a.$$

Furthermore, for $n=2$, equation (8) defines the cosine curve

$$(8'')\qquad y=a\sqrt{1-u^2},\qquad ct=a\sin^{-1}u;\quad \text{or,}\quad y=a\cos(ct/a).$$

In general, for positive n, if we set $n=2/p$, $u=\sqrt{p}\sin\phi$, we obtain

$$(9)\quad y=a\cos^p\phi,\quad ct=a\sqrt{p}\int_0^\phi\cos^{p-1}\phi d\phi,\quad v=-c\sqrt{p}\sin\phi,\quad p=2/n.$$

In general, for negative n, if we set $n=-2/m$, $u=\sqrt{m}\tan\phi$, we obtain

$$(9') \quad y = a\cos^m\phi, \quad ct = a\sqrt{m}\int_0^\phi \cos^m\phi\, d\phi, \quad v = -c\sqrt{m}\tan\phi, \quad m = -2/n.$$

For $n=0$, the expressions in (8) must be looked upon as indeterminate forms whose limits for $n\to 0$ are given by (7').

5. The total time. The total time T required for the trip from $y=a$ to $y=0$ is given by definite integrals which may be evaluated in terms of the Beta and Gamma functions. For $n=2/p>0$ we have

$$(10) \quad \begin{aligned} T &= \frac{a\sqrt{p}}{c}\int_0^{\pi/2} \cos^{p-1}\phi\, d\phi = \frac{a\sqrt{p}}{c2} B\left(\frac{p}{2}, \frac{1}{2}\right) \\ &= \frac{a}{c}\sqrt{\frac{\pi}{2}}\sqrt{\frac{p}{2}}\,\Gamma\left(\frac{p}{2}\right)\Big/\Gamma\left(\frac{p+1}{2}\right), \quad \frac{p}{2} = \frac{1}{n}. \end{aligned}$$

Similarly for $n=-2/m<0$, we have

$$(10') \quad \begin{aligned} T &= \frac{a\sqrt{m}}{c}\int_0^{\pi/2} \cos^m\phi\, d\phi = \frac{a\sqrt{m}}{c2} B\left(\frac{m+1}{2}, \frac{1}{2}\right) \\ &= \frac{a}{c}\sqrt{\frac{\pi}{2}}\,\Gamma\left(\frac{m+1}{2}\right)\Big/\sqrt{\frac{m}{2}}\,\Gamma\left(\frac{m}{2}\right), \quad \frac{m}{2} = -\frac{1}{n}. \end{aligned}$$

These two results may be combined in a single formula

$$(11) \quad T = \frac{a}{c}\sqrt{\frac{\pi}{2}}\, e^{f(n)},$$

by means of the odd function $f(n)$, which we define by the equations

$$(12) \quad \begin{aligned} f(n) &= \ln\sqrt{\frac{1}{n}}\,\Gamma\left(\frac{1}{n}\right) - \ln\Gamma\left(\frac{1}{n}+\frac{1}{2}\right), \qquad n>0; \\ f(0) &= 0; \qquad f(-n) = -f(n). \end{aligned}$$

It can be shown that $f(n)$ has the power series expansion

$$(13) \quad f(n) = \frac{n}{8} - \frac{n^3}{192} + \frac{n^5}{640} - \cdots = \sum_{k=1}^{\infty} \frac{(-1)^k(1-4^k)B_k}{(2k-1)(2k)}\left(\frac{n}{2}\right)^{2k-1},$$

where B_k are the Bernoulli numbers ($B_1=1/6$, $B_2=1/30$, $B_3=1/42$, *etc*). It may be an interesting exercise to evaluate three terms of the series for $n=\pm 1$, ± 2, and compare (11) with the values for T in (10) and (10').

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

The Education of T. C. Mits. By H. G. Lieber and L. R. Lieber. New York, W. W. Norton and Company, Inc., 1944. 230 pages. $2.50.

This book, as the title indicates, is concerned with the education of "The Celebrated Man In the Street." It is "an attempt to get a bird's eye view of T. C.'s predicament, and to look for a possible egress." To do this vividly, the author uses pictures whenever possible, and to do it clearly, he uses the clearest language man has invented, mathematics.

The contents of the book are divided into two parts. Part I is what the author calls *The Old*. Among other things he points out by examples from mathematics, that the average person should not depend on intuition to draw conclusions; that, as in algebra, the use of generalization is a distinct advantage; that, as in geometry, abstraction is a powerful tool; that the mathematician by means of using "the marriage of algebra and geometry," analytic geometry, and its "offspring," calculus, can easily study our ever changing world; and most important, that science and mathematics can not only protect us from actual physical dangers, but also from the errors of our loose thinking "and thus be a veritable defense against all evil—a Totem Pole." The Totem Pole contains five divisions, from the bottom floor where all the scientific gadgets are kept, to the top floor where the pure mathematicians dwell. The author draws the conclusion in his summary of Part I "that mathematics is not only for someone who needs its formulas. It is a way of thinking, a way of life, very important for everyone."

Part II is what the author calls *The New*. In this section the author points out how the study of any science or system of thought can be developed starting with a few basic ideas from which all the other ideas or "propositions" are derived by logic. By giving several examples from mathematics, he presents the pitfalls that may lie ahead, those of using guess work, limited knowledge, and not keeping an open-minded attitude. Man must use his own reason to the best of his ability and he will obtain respectable results, but he must not brag that he "knows" the truth. As in mathematics, future observations may change the present. T. C. Mits must adapt himself to a continually changing world.

The reviewer would like to picture the book as an excellent attempt to popularize mathematics for ordinary people, an amusing work, and yet full of a deep philosophy of life. It is presented in a unique and interesting fashion, greatly aided by the illustrations. The book deserves greater popularity with the non-mathematician as well as the mathematician.

E. P. VANCE

Algebra of Analysis. By Karl Menger. Notre Dame, Indiana (Notre Dame Mathematical Lectures, Number 3), University of Notre Dame, 1944. 3+50 pages. $1.00.

The author takes a new approach to the theory of functions of a single variable by starting with a system of elements (to be called functions) which form a tri-operational algebra. The operations of this algebra of functions are addition, multiplication and substitution. The commutative, associative and distributive laws for these operations which are satisfied by ordinary functions are postulated. The author defines neutral or identity elements in regard to each operation, the neutral element for the substitution operation corresponding to the function $y=x$ in the ordinary function theory.

Topics covered include the theory of constant functions, the theory of inverses (where these exist) for the three operations of the algebra of functions, definitions and properties of exponential, logarithmic, power and trigonometric functions, the algebra of calculus including anti-derivatives. The last chapter explores the concept of algebra of functions of several variables.

The booklet is written in a fluent style and it can be readily appreciated by a reader who has an elementary knowledge of the concepts of ring and field. The limited distributivity available in regard to the substitution operation impedes the development of any very deep algebraic theory for the algebra of functions. In some topics, the author's notation and method achieves considerable elegance; this is, however, balanced on occasion by formalism and expedient postulation.

A number of minor errors were observed by the reviewer. In the footnote on p. 11, third line from the bottom, "and $=0$" should read "and $=c$." On p. 16 the formulas for j.rec and f.rec f should have 1 not j as their right members. On p. 19 the discussion of exponential functions where the constants form a finite field requires slight correction. On page 31, line 7, $(Dg)f$ should be $(Df)g$, and on page 32, line 2, the left member should be D neg j.

C. J. NESBITT

Mathematics. Second Edition. By J. W. Breneman. New York and London, McGraw-Hill Book Company, Inc., 1944. 12+224 pages. $1.75.

This text covers arithmetic, algebra to the solution of quadratics, geometrical constructions without proofs, trigonometry, and logarithms in an elementary and practical manner. It contains nearly five hundred simple and practical problems taken from engineering applications whenever possible.

The emphasis on theory varies from no proof in geometry to careful proofs of the sine and cosine laws. The laws of logarithms are presented by simple numerical examples but no formal proofs are given.

Besides the usual tables the appendix contains some tables ordinarily placed in engineering texts and a splendid table of specific gravities and weights of common substances.

R. D. WAGNER

Air Navigation Made Easy. By J. F. Naidich. New York and London, McGraw-Hill Book Company, Inc., 1944. 9+124 pages. $1.75.

This book is intended for private civilian flyers who plan flights under three hundred or four hundred miles and who have had very little technical training. It covers the subjects of piloting, *i.e.*, flying by use of land marks; dead reckoning, *i.e.*, flying by calculated data; and, briefly, flying by radio aids. It contains numerous questions and examples to fix in mind the principles that the flyer should know thoroughly. Answers to the odd-numbered questions and examples follow the main body of the text.

The book is divided into four parts. Part 1, *Air Piloting*, consists of one chapter on maps and charts, and their uses in air piloting. Mercator, Lambert, and polar projections are explained. Part 2, *Dead Reckoning*, is divided into two chapters. Chapter II deals with the aircraft compass, its errors, and use in flying. Chapter III is on the wind triangle, the solutions of which are obtained graphically. Part 3, *Locating Position and Other Problems*, has two chapters. Locating position while in flight is given in Chapter IV. Flying time, range, radius of action, and a brief discussion of Federal Aids to Navigation are given in Chapter V. Part 4, *Review Tests*, has five tests covering the main points in the other three parts.

The reviewer found no errors. The mechanical make-up of the book is excellent. All printing is clear and distinct. The author's style of presentation is good and one is able to read without tiring. The only adverse criticism the reviewer is inclined to make is concerning the expression on page 1 about the "top" of the earth, when the polar region is meant.

Anyone learning to fly, whether he be preparing to fly for pleasure or preparing to operate a plane for other reasons, will find this a very helpful book.

H. H. DOWNING

Vital Mathematics. By E. B. Allen, Dis Maly, and S. H. Starkey, Jr. New York, The Macmillan Company, 1944. 7+456+22 pages. $1.80.

This text covers arithmetic, algebra through quadratic equations, plane geometry without formal proofs, some solid geometry, plane trigonometry, and an introduction to spherical trigonometry. The simple operations of arithmetic are carefully presented. The theory and the practice of rounding off numbers are thoroughly discussed.

There is a large collection of examples and interesting problems. Many of them emphasize practical applications of mathematics.

Each chapter is headed by a set of problems typical of those occurring in the chapter. This makes the text especially adaptable for self study.

Some unusual materials included are elementary statistics and an introduction to cartography.

R. D. WAGNER

Plane and Spherical Trigonometry. By H. P. Doole. New York, Thomas Crowell Co., 1944. 8+183 pages. $1.75.

This rearrangement of the standard materials of trigonometry is intended for the instructor who wishes to complete the theory before turning to the applications. For example, radian measure and inverse functions appear in the first lessons. Three chapters present theory (17 exercises, 569 problems); two chapters are devoted to logarithms and the solution of triangles (14, 232); there are chapters on graphs, mil measure, slide rule, and vectors (5, 88); and a chapter on spherical trigonometry (8, 74). A companion handbook of tables is required, only two brief tables being given in the text. An index and a list of answers to odd-numbered problems are included.

The volume is a handy size, all figures are clear, and there are stimulating drawings as chapter headings. Brief statements are given of the use of each new concept, and explanations are brief and simple. It may be valuable in teaching identities to use the author's symbols A and T to distinguish those steps of a proof which are algebraic and those which are trigonometric.

The text assumes that the student knows the rectangular coördinate system. There is little review of the material from solid geometry used in spherical trigonometry. Proofs for some identities are not extended to all cases. The term "distance" is introduced for "radius vector." The term "knots per hour" is used.

Doole's text will be found suitable, subject to the comments above, for either a brief or an extended course.

B. M. STEWART

New List of Mathematical Tables. Washington, D. C., National Bureau of Standards, 1945. 14 pages. No charge.

The following paragraphs are quoted from a communication recently received from the National Bureau of Standards:

"A list of the 48 mathematical tables which have thus far been prepared by the Mathematical Tables Project and made available to the public is given in Letter Circular LC-777, just issued by the National Bureau of Standards.

"31 of these tables may be purchased from the Bureau, 2 from the Government Printing Office, 4 from the Columbia University Press, and nearly all of the remainder can be consulted in the several mathematical journals referred to.

"The Project was conducted by the Work Projects Administration for the City of New York until March 1943, at which time the Bureau (the sponsoring agency) took over its operation with the support of the Office of Scientific Research and Development. As time permits, various tables under way when the WPA was discontinued, will be completed.

"Copies of Letter Circular LC-777, 'Mathematical Tables,' may be had by writing to the Information Section, National Bureau of Standards, Washington 25, D. C."

H. P. EVANS

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solution to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 671. *Proposed by Alice Malice, Owen Sound, Ontario*

A hobo has crossed three-eighths of a high railway bridge when he hears behind him an express train coming at sixty miles per hour. He can just save his life by running to either end of the bridge. How fast can he run? (There is no question of acceleration or deceleration; the engineer is merciless and the man gets going at once.)

E 672. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the cyclic numbers in the duodenary scale. (A number is said to be *cyclic* if every cyclic permutation of its digits produces a multiple of the number.)

E 673. *Proposed by L. S. Shively, Ball State Teachers College*

Does there exist a regular polygon having both these properties: (a) a diagonal is equal to the sum of two other diagonals; (b) a diagonal is equal to the sum of a side and another diagonal?

E 674. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that a pandiagonal magic square of order 4, when regarded as a determinant, has the value zero.

E 675. *Proposed by Howard Eves, Syracuse University*

Show that the ratio of the curvatures of two curves in a plane at a point of contact is invariant under projection.

SOLUTIONS

An Infinite Matrix

E 634 [1944, 405]. *Proposed by H. S. Wall, Northwestern University*

Let a matrix be constructed according to the following rules:

$$a_{1,1} = 1, \qquad a_{p,q} = 0 \quad \text{if} \quad p < q,$$

$$a_{p,1} = a_{p-1,2} \qquad (p = 2, 3, \cdots),$$

$$a_{p,q} = a_{p-1,q-1} + a_{p-1,q+1} \qquad (p, q = 2, 3, \cdots).$$

Show that the sum of the products of corresponding elements in any two rows is equal to an element in the first column, *viz.*,

$$\sum_{r=1}^{\infty} a_{p,r}\, a_{q,r} = a_{p+q-1,1} \qquad (p, q = 1, 2, \cdots).$$

I. *Solution by N. J. Fine, Naval Ordnance Plant, Indianapolis.* Define $\|b_{p,q}\|$ by the equation

$$a_{p,1} = \sum_{q=1}^{\infty} b_{k,q}\, a_{p-k,q} \qquad (k = 0, 1, 2, \cdots).$$

Clearly $b_{0,1}=1$, and $b_{k,q}=0$ if $q>k+1$. Using the recursion formulas for the $a_{p,q}$,

$$a_{p,1} = b_{k,1}\, a_{p-k-1,2} + \sum_{q=2}^{\infty} b_{k,q}(a_{p-k-1,q-1} + a_{p-k-1,q+1})$$

$$= b_{k,1}\, a_{p-k-1,1} + \sum_{q=2}^{\infty} (b_{k,q-1} + b_{k,q+1}) a_{p-k-1,q};$$

but also

$$a_{p,1} = \sum_{q=1}^{\infty} b_{k+1,q}\, a_{p-k-1,q}.$$

Hence

$$b_{k+1,1} = b_{k,2} \qquad (k = 0, 1, \cdots),$$

$$b_{k+1,q} = b_{k,q-1} + b_{k,q+1} \qquad (k = 0, 1, \cdots; q = 2, 3, \cdots).$$

Since the matrix $\|b_{p-1,q}\|$ has the same initial conditions and the same recursion formulas as $\|a_{p,q}\|$, the two are identical, that is,

$$b_{p-1,q} = a_{p,q} \qquad (p, q = 1, 2, \cdots).$$

Hence

$$a_{p,1} = \sum_{r=1}^{\infty} a_{k+1,r}\, a_{p-k,r} \qquad (k = 0, 1, \cdots).$$

Setting $P=k+1$, $Q=p-k$, so that $p=P+Q-1$, we have, finally,

$$a_{P+Q-1,1} = \sum_{r=1}^{\infty} a_{P,r}\, a_{Q,r} \qquad (P, Q = 1, 2, \cdots).$$

II. *Solution by the Proposer.* Define $a_{p,0}=0$ and

$$C_{p,q} = \sum_{r=1}^{\infty} a_{p,r}\, a_{q,r} \qquad (p, q = 1, 2, \cdots).$$

Then

$$C_{p,q+1} = \sum_{r=1}^{\infty} a_{p,r}\, a_{q+1,r} = \sum_{r=1}^{\infty} a_{p,r}(a_{q,r-1} + a_{q,r+1})$$

$$= \sum_{r=1}^{\infty} a_{p,r+1}\, a_{q,r} + \sum_{r=1}^{\infty} a_{p,r}\, a_{q,r+1},$$

which involves p and q symmetrically, so that

$$C_{p,q+1} = C_{p+1,q}$$

$$= C_{p+2,q-1} = \cdots = C_{p+q,1}.$$

Hence

$$C_{p,q} = C_{p+q-1,1} = \sum_{r=1}^{\infty} a_{p+q-1,r}\, a_{1,r} = a_{p+q-1,1}.$$

Also solved by E. P. Starke, who points out that

$$a_{p,q} = \binom{p-1}{\left[\frac{p-q}{2}\right]} - \binom{p-1}{\left[\frac{p-q-1}{2}\right]}.$$

The Linking Rings

E 636 [1944, 472]. *Proposed by P. R. Halmos, Syracuse University*

What is the least number of links that have to be cut, in a chain of n links, so that every integer between 1 and n can be represented as a sum of numbers of links in the disconnected chains obtained? (Cutting one link which is not at either end of a connected chain produces three pieces: the severed link and the two ends which it held together.)

Solution by Monte Dernham, San Francisco. Assuming an indefinitely extended chain, let k links be cut, so that the maximum number of successive integers, say m, can be represented in the manner described. The severed links permit representation of every integer from 1 to k. To represent the next integer, $k+1$, most economically, one of the disconnected pieces must contain $k+1$ links. This piece, in combination with one or more of the severed links, further permits representation of each of the succeeding k integers. Continuing in like manner, we find that additional segments, if any, must contain respectively

$$2(k+1), 4(k+1), \cdots, 2^r(k+1), \cdots$$

links; and, since the number of disconnected segments which the k severed links held together cannot exceed $k+1$,

$$m = k + \sum_{r=0}^{k} 2^r(k+1) = 2^{k+1}(k+1) - 1.$$

It follows that, in a chain of n links, the least number of links that have to be cut is the integer k for which

$$2^k k \leqq n < 2^{k+1}(k+1).$$

Thus $k=1$ for $2 \leqq n \leqq 7$,
$k=2$ for $8 \leqq n \leqq 23$,
$k=3$ for $24 \leqq n \leqq 63$,
$k=4$ for $64 \leqq n \leqq 159$, and so on.

Finally, it may be observed that there are ways of producing disconnected chains, and reconnecting them, without cutting any links whatsoever—exemplified by such finished artists as Dante, the Great Leon, Jean Hugard, Dr. Harlan Tarbell, Chung Ling Soo. Monographs on this effect appear occasionally in *The Linking Ring*, official journal of the International Brotherhood of Magicians. But extreme care must be taken to preclude interference by disembodied spirits; otherwise the chains are apt to become unmanageable.

Also solved by Shepard Bartnoff, D. H. Browne, W. E. Buker, E. D. Schell, E. P. Starke, and the proposer.

Seven Sevens at the End of a Cube

E 638 [1944, 472]. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

Without the use of tables, find the smallest integer whose cube terminates in seven sevens.

Solution by L. R. Chase, Rogers High School, Newport, R. I. Let a denote an n-digit number, such that a^3 ends in n sevens preceded by a digit y. Let a new digit x be prefixed to a to form an $(n+1)$-digit number $x \cdot 10^n + a$, whose cube is

$$x^3 \cdot 10^{3n} + 3ax^2 \cdot 10^{2n} + 3a^2x \cdot 10^n + a^3.$$

Here x may be selected so that the number formed by the last two terms shall end in $n+1$ sevens; the other terms cannot affect the last $n+1$ digits. Since $3a^2 \equiv 147 \equiv 7 \pmod{10}$, these last two terms are

$$7x \cdot 10^n + a^3 \pmod{10^{n+1}}.$$

Hence we require $7x+y \equiv 7 \pmod{10}$, *i.e.*,

$$x \equiv 7y + 1 \pmod{10}.$$

By trial we find, when $n=1$, $a=3$, so that $a^3=27$, $y=2$, $x=5$.

When $n=2$, we have	$a=$ 53,	$a^3=$ $\cdots$877,	$y=8$, $x=7$.
When $n=3$,	$a=$ 753,	$a^3=$ $\cdots$7777,	$y=7$, $x=0$.
When $n=4$,	$a=$ 0753,	$a^3=$ $\cdots$57777,	$y=5$, $x=6$.
When $n=5$,	$a=$ 60753,	$a^3=$ $\cdots$577777,	$y=5$, $x=6$.
When $n=6$,	$a=$ 660753,	$a^3=$ $\cdots$4777777,	$y=4$, $x=9$.
When $n=7$,	$a=9660753$.		

Also solved by Shepard Bartnoff, Colin Blyth, D. H. Browne, W. E. Buker, F. M. Carpenter, Monte Dernham, William Douglas, Margaret Olmsted, E. D. Schell, E. P. Starke, J. A. Tierney, Jeanette Van Os, and the proposer.

The Pole of a Clothoid

E 639 [1944, 472]. *Proposed by Howard Eves, Syracuse University*

A *clothoid* (or *transition spiral*, used in highway engineering) is defined as a curve whose curvature varies directly with the arc length. Locate the geometrical pole of this spiral.

Solution by the Proposer. Let the start O of the spiral be taken as origin of rectangular coordinates, and let the initial tangent OX be taken as the x-axis. Then, if s is the arc length of the spiral from O to any point P on the curve, and ϕ the angle which the tangent at P makes with OX, we have, by the definition of the curve,

$$d\phi/ds = ks,$$

where k is a constant of proportionality. Since $s=0$ when $\phi=0$, it follows that

$$s = 2K\phi^{1/2},$$

where $K=(2k)^{-1/2}$. Hence

$$dx = ds\cos\phi = K\phi^{-1/2}\cos\phi d\phi, \qquad dy = ds\sin\phi = K\phi^{-1/2}\sin\phi d\phi,$$

and

$$x = K\int_0^{\phi}\phi^{-1/2}\cos\phi d\phi, \qquad y = K\int_0^{\phi}\phi^{-1/2}\sin\phi d\phi.$$

But the pole of the spiral corresponds to $\phi=\infty$, and

$$\int_0^{\infty}\phi^{-1/2}\cos\phi d\phi = \int_0^{\infty}\phi^{-1/2}\sin\phi d\phi = \sqrt{\pi/2}.$$

Therefore the pole is located at the point $(K\sqrt{\pi/2}, K\sqrt{\pi/2})$, or, in terms of the original constant, $(\frac{1}{2}\sqrt{\pi/k}, \frac{1}{2}\sqrt{\pi/k})$.

An Exponential Equation

E 640 [1944, 472]. *Proposed by E. D. Schell, Arlington, Virginia*

Solve in integers the equation

$$x^y = y^x \qquad (x > y).$$

Solution by M. D. D. Burrow, McGill University. Writing the equation in the form $x^{1/x}=y^{1/y}$, we see that the problem reduces to finding integral values of x which give coincident values of

$$f(x) = x^{1/x}.$$

Now, $f(x)$ increases for $0<x<e$, and then decreases but remains greater than 1. Thus the smaller value of x, if positive, must lie between 1 and e, where the only integer is 2. Hence the only positive integral solution is $y=2$, $x=4$.

Also solved by Shepard Bartnoff, Colin Blyth, D. H. Browne, W. E. Buker, L. H. Bunyan, H. N. Carleton, L. R. Chase, Monte Dernham, Howard Eves, Daniel Finkel, L. R. Ford, Irving Kaplansky, J. B. Kelly, Samuel Kramer, C. D. Olds, E. S. Pondiczery, Alexander Russ, E. P. Starke, Martha Sved, Jeanette Van Os, and the proposer.

Many (like Burrow and the proposer) overlooked the second solution

$$x = -2, \qquad y = -4.$$

Buker and Olds pointed out that Euler solved the given equation in the form

$$x = \left(1+\frac{1}{n}\right)^{n+1}, \qquad y = \left(1+\frac{1}{n}\right)^{n},$$

and by D. Bernoulli in the equivalent form $x=c^{c/(c-1)}$, $y=c^{1/(c-1)}$. Here $n=1/(c-1)$ and $c=\log_y x$. For a simple proof that Euler's expressions, for integers n, are the only positive rational solutions, see R. L. Goodstein, *Mathematical Gazette*, vol. 28 (May 1944), p. 76. See also this MONTHLY, vol. 28 (1921), pp. 141–143, and vol. 38 (1931), pp. 444–447.

Swimming Around a Lighter

E 641 [1944, 530]. *Proposed by F. M. Garnett, Savannah, Ga.*

A lighter, twenty by thirty feet, travels upstream at a uniform speed of 2 m.p.h., against a current of 1½ m.p.h. A swimmer, whose rate of swimming is 4 m.p.h. in still water, swims around the lighter. How long does he take to complete the circuit?

I. *Solution by Frank Hawthorne, Allegheny College.* It is assumed that the swimmer starts and finishes his circuit at the same position relative to the lighter, and that the given speed of 2 m.p.h. is relative to the land.

The man's net speed, relative to the lighter, while swimming up one side and down the other, will be ½ and 7½ m.p.h., or 11/15 and 11 f.p.s. His time for the two trips together is thus

$$30\left(\frac{15}{11}+\frac{1}{11}\right)=\frac{480}{11}$$

seconds. In crossing the bow and stern he must maintain a forward component of 3½ m.p.h. relative to the water; so his transverse component is

$$\sqrt{4^2-(3\tfrac{1}{2})^2}=\tfrac{1}{2}\sqrt{15}$$

m.p.h., or $11/\sqrt{15}$ f.p.s. Since he travels a total distance of 40 feet transversely his time for these parts of the circuit is $40\cdot\sqrt{15}/11$ seconds. His total time is thus

$$\frac{480}{11}+\frac{40\sqrt{15}}{11}=\frac{40(12+\sqrt{15})}{11}=57.72$$

seconds.

Also solved thus by Murray Barbour, W. E. Buker, Leon Bunyan, Monte Dernham, Howard Eves, E. P. Starke and W. Unterberg. *Cf.* problem 3218 [1927, 388].

II. *Solution by D. H. Browne, Buffalo, N. Y.* Because of the pressure wave at the bow and the drag at the stern, the swimmer is virtually in still water at either end. (I incline to this simpler view, having experienced it!) He passes the length of the boat at $\frac{1}{2}$ m.p.h. upstream and $7\frac{1}{2}$ m.p.h. downstream, covering the transverses at 4 m.p.h. and making the total encirclement in

$$\frac{2}{44}+\frac{30}{44}+\frac{5}{44}=\frac{37}{44}$$

minutes, or $50\frac{5}{11}$ seconds.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4158. *Proposed by P. D. Thomas, Lumberton, Miss.*

Integrate the partial differential equation

$$(1+q^2)r-2pqs+(1+p^2)t+(rt-s^2)/(1+p^2+q^2)^{1/2}+(1+p^2+q^2)^{3/2}=0;$$

and give a geometrical interpretation of the general integral.

4159. *Proposed by F. J. Duarte, Caracas, Venezuela*

Given the 18 numbers a_i, b_i, c_i, $i=1, 2, 3, 4, 5, 6$ such that

$$\begin{vmatrix} (bc)_{12} & (ca)_{12} & (ab)_{12} \\ (bc)_{25} & (ca)_{25} & (ab)_{25} \\ (bc)_{46} & (ca)_{46} & (ab)_{46} \end{vmatrix} = \begin{vmatrix} (bc)_{24} & (ca)_{24} & (ab)_{24} \\ (bc)_{15} & (ca)_{15} & (ab)_{15} \\ (bc)_{36} & (ca)_{36} & (ab)_{36} \end{vmatrix} = 0$$

where

$$(mn)_{ij}=\begin{vmatrix} m_i & m_j \\ n_i & n_j \end{vmatrix},$$

show that

$$\begin{vmatrix} (bc)_{12} & (ca)_{12} & (ab)_{12} \\ (bc)_{34} & (ca)_{34} & (ab)_{34} \\ (bc)_{56} & (ca)_{56} & (ab)_{56} \end{vmatrix} = 0.$$

4160. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show how to construct four spheres passing through a given point and tangent respectively to the planes of three faces of a given tetrahedron so that the points of contact are twelve points of the same sphere.

SOLUTIONS

Factorial Coefficients

4108 [1944, 96]. *Proposed by G. Pólya, Stanford University*

Let ${}_nP_r$ be the number of those permutations of n elements which are the products of exactly r cycles without common elements. For instance, ${}_4P_2=11$. Let ${}_nQ_r$ be the number of different classifications of n distinct elements into exactly r classes. For instance, ${}_4Q_2=7$. Prove that

$$(1) \qquad {}_nP_1x + {}_nP_2x^2 + \cdots + {}_nP_nx^n = x(1+x)(2+x)\cdots(n-1+x),$$

$$(2) \qquad {}_nQ_1x + {}_nQ_2x(x-1) + \cdots + {}_nQ_nx(x-1)\cdots(x-n+1) = x^n.$$

Note. In the Calculus of Finite Differences, ${}_nP_r$ and ${}_nQ_r$ are considered in various notations and are often called factorial coefficients; see, for instance, Steffensen, *Interpolation*, Baltimore, 1927, pp. 53–58.

Solution by P. R. Halmos, Syracuse University. A permutation of $n+1$ elements which is the product of exactly r cycles may be obtained in one of two ways from permutations of n elements. The first way is to adjoin a cycle of length one involving a new element to a permutation which is the product of $r-1$ cycles; the second way is to adjoin a new element to any cycle of a permutation which is the product of r cycles. Since in the second operation the new element may be inserted after any one of the r old elements, we see that

$${}_{n+1}P_r = {}_nP_{r-1} + n\,{}_nP_r.$$

(If ${}_nP_0$ is interpreted to be 0, this equation is valid for $n=1, 2, \cdots, r=1, \cdots, n$.) If we write $p_n(x)=\sum_{r=1}^{n} {}_nP_rx^r$ then the relation between the P's implies that $(x+n)p_n(x)=p_{n+1}(x)$, from which (1) follows immediately by mathematical induction.

Similarly, a classification of $n+1$ elements into r classes may be obtained in one of two ways from classifications of n elements. The first way is to adjoin a class consisting of exactly one new element to a classification into $r-1$ classes; the second way is to adjoin a new element to any class of a classification into r classes. Since in the second operation the new element may be inserted into any one of the r existing classes, we see that

$$ {}_{n+1}Q_r = {}_nQ_{r-1} + r\,{}_nQ_r. $$

(If ${}_nQ_r$ is interpreted to be 0 unless $1 \leqq r \leqq n$, this equation is valid for $n, r = 1, 2, \cdots$). Write now $f_r(x) = x(x-1)\cdots(x-r+1)$ and $q_n(x) = \sum_{r=1}^{n} {}_nQ_r f_r(x)$. Since, as is easily verified, $f_{r+1}(x) + rf_r(x) = xf_r(x)$, (2) follows inductively from the equations

$$ q_{n+1}(x) = \sum_{r=1}^{n+1} {}_{n+1}Q_r f_r(x) = \sum_{r=1}^{n+1} ({}_nQ_{r-1} + r\,{}_nQ_r) f_r(x) $$

$$ = \sum_{r=1}^{n} {}_nQ_r(f_{r+1}(x) + rf_r(x)) = x\sum_{r=1}^{n} {}_nQ_r f_r(x) = xq_n(x). $$

Solved also by H. W. Becker, G. F. Morecroft, C. D. Olds, and Louis Weisner.

Editorial Note. The solution by Morecroft is similar to the above. Olds stated that $P = m!\,{}_nQ_m/m^n$ is the probability that, in n drawings of a single number from an urn containing the numbers $1, 2, \cdots, m$, replacing the number in the urn before the next drawing, each number will be drawn at least once. He referred to Ch. Jordan's *Calculus of Finite Differences*, Budapest, 1939, p. 604. Weisner used in his proofs the formulas

$$ {}_nP_r = \sum_{\alpha} \frac{n!}{\alpha_1!\alpha_2!\cdots\alpha_n!\,2^{\alpha_2}3^{\alpha_3}\cdots n^{\alpha_n}}, $$

$$ {}_nQ_r = \sum_{\alpha} \frac{n!}{\alpha_1!\alpha_2!\cdots\alpha_n!\,(2!)^{\alpha_2}(3!)^{\alpha_3}\cdots(n!)^{\alpha_n}}, $$

$$ \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n, \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_n = r $$

where the α_i's are non-negative integers. Becker gave interesting remarks with references to E 461 [1941, 701], E 565 [1944, 47], Riordan, *Moment Recurrence Relations*, Ann. Math. Stat. VIII, 103; Steffensen, *Statistics and Actuarial Science*, Cambridge, 1930, 24; E. T. Bell, *Generalized Stirling Transforms*, Am. Jour. Math. 61, 1939, 89, and *Iterated Exponential Integers*, Annals of Math. 39, 1938, 539.

The equation (2) of the problem may be written

$$ x^n = \sum_{r=1}^{n} {}_nQ_r x^{(r)}, \qquad x^{(r)} = x(x-1)\cdots(x-r+1), \qquad x^{(0)} = 1, $$

and the coefficients ${}_nQ_r$ can be computed by a process similar to the synthetic division method for reducing the roots of a polynomial by a constant a. Here in the first division we use $a = 1$, in the second $a = 2$, and so on. For $n = 5$ the divisions are

	1		0		0		0		0
			1		1		1		1
${}_1Q_1$	1	${}_2Q_1$	1	${}_3Q_1$	1	${}_4Q_1$	1	${}_5Q_1$	1
			2		6		14		
${}_2Q_2$	1	${}_3Q_2$	3	${}_4Q_2$	7	${}_5Q_2$	15		
			3		18				
${}_3Q_3$	1	${}_4Q_3$	6	${}_5Q_3$	25				
			4						
${}_4Q_4$	1	${}_5Q_4$	10						
${}_5Q_5$	1								

$$x^5 = x^{(5)} + 10x^{(4)} + 25x^{(3)} + 15x^{(2)} + x^{(1)}.$$

For a larger table we extend the rows and columns and for each value of n we read the results along the secondary diagonal from ${}_nQ_nn^{(n)}$ to ${}_nQ_1x^{(1)}$. From two consecutive rows we have

$$\begin{array}{lll} \cdots\ {}_{n-1}Q_{r-1} & {}_nQ_{r-1} & (r-1) \\ \cdots\ r{}_{n-1}Q_r & r{}_nQ_r & \\ \hline \cdots\ {}_nQ_r & {}_{n+1}Q_r & (r) \end{array}$$

and thus ${}_{n+1}Q_r = {}_nQ_{r-1} + r{}_nQ_r$. Since $\Delta x^{(r)} = rx^{(r-1)}$, we have

$$\Delta^r x^n = \sum_{j=r}^{n} {}_nQ_j j^{(r)} x^{(j-r)},$$

and it follows that $\Delta^r 0^n = r!{}_nQ_r$, ${}_nQ_r = \Delta^r 0^n / r!$ The above expression for x^5 may be written

$$x^5 = \frac{\Delta x^{(6)}}{6} + 10\frac{\Delta x^{(5)}}{5} + 25\frac{\Delta x^{(4)}}{4} + 15\frac{\Delta x^{(3)}}{3} + \frac{\Delta x^{(2)}}{2}.$$

Hence

$$\sum_{x=1}^{n-1} x^5 = \frac{n^{(6)}}{6} + 10\frac{n^{(5)}}{5} + 25\frac{n^{(4)}}{4} + 15\frac{n^{(3)}}{3} + \frac{n^{(2)}}{2}.$$

From (1) in the problem it is seen that ${}_nP_{n-r} = \sigma_r(n)$, the rth elementary symmetric function of $1, 2, \cdots, n-1$ which is considered in 3940 [1941, 641]. It was shown that

$$\sigma_r(n) = \sum_{t=1}^{r} b_t^r n^{(r+t)},$$

$$\sigma_r(n+1) = \sigma_r(n) + n\sigma_{r-1}(n), \qquad \sigma_0(n) = 1,$$

$$b_1^r = \frac{1}{r+1}, \qquad b_r^r = \frac{1}{r!2^r}, \qquad b_t^r = \frac{b_{t-1}^{r-1} + (r+t-1)b_t^{r-1}}{r+t}.$$

The values of b_t^r are there given for $0 \leqq r \leqq 5$. A variation of the method used gives also

$$\sigma_r(n) = \sum_{t=1}^{r} (-1)^{r+t} a_t^r (n+t-1)^{(r+t)};$$

$$a_1^r = \frac{1}{(r+1)!}, \qquad a_r^r = \frac{1}{r!2^r}, \qquad a_t^r = \frac{a_{t-1}^{r-1} + t a_t^{r-1}}{r+t};$$

$$a_2^2 = \frac{1}{12}, \qquad a_2^4 = \frac{5}{144}, \qquad a_3^4 = \frac{1}{48}, \qquad a_2^5 = \frac{1}{90},$$

$$a_3^5 = \frac{7}{(4!)^2}, \qquad a_4^5 = \frac{1}{2!3!4!}.$$

We shall now show that

$$\sigma_r(-n) = {}_{n+r}Q_n = \frac{\Delta^n 0^{n+r}}{n!}.$$

We may write the difference equations in the forms

$$\sigma_r(-n) = \sigma_r[-(n-1)] + n\sigma_{r-1}(-n)$$

$${}_{n+r}Q_n = {}_{(n-1)+r}Q_{n-1} + n_{n+(r-1)}Q_n$$

which are of the form $f_r(n) = f_r(n-1) + nf_{r-1}(n)$, for $r = 0$ in the two cases we have $\sigma_0(-n) = {}_nQ_n = 1$, and this suffices to prove the desired result. Thus

$${}_nQ_{n-r} = \sum_{t=1}^{r} a_t^r n^{(r+t)} = \sum_{t=1}^{r} (-1)^{r+t} b_t^r (n+t-1)^{(r+t)}.$$

Sum of Two Squares Equal to a Square

4110 [1944, 96]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the smallest integer whose square is the sum of two squares in seven different ways.

I. *Solution by Colin Blyth, Queen's University Student.* The following theorems are used:

1. If $p = 4n+1$, a prime, then p^2 is the sum of two squares in exactly one way. If $p = 4n+3$, a prime, then it is impossible to express p^2 as the sum of two squares.

2. If $N = m\prod_{i=1}^{n} p_i^{a_i}$ where every prime factor of m is of the form $4k+3$ or 2,

and the p_i are distinct primes of the form $4k+1$, then N^2 is the sum of two squares in the same number of ways as is $\prod_{i=1}^{r} p_i^{a_i}$, namely

$$2^{n-1}+\sum \text{ Number of ways for every possible factor.}$$

Since we are looking for the smallest integer we can find, we must obviously take $m=1$.

The first few primes of type $4k+1$ are:

$$5,\ 13,\ 17,\ 29,\ 37,\ 41,\ \cdots.$$

By making a few trials we quickly arrive at the required integer:

5^2 is expressible in 1 way

$(5.5)^2$ is expressible in $1+1=2$ ways

$(5.13)^2$ is expressible in $2+1+1=4$ ways

. . . . . .

$(5.5.13)^2$ is expressible in $2+1+2+1+1=7$ ways.

This solution does not provide a means of finding these actual sums. The seven ways are readily found to be:

$$\begin{aligned}(5.5.13)^2 &= 204^2+253^2\\ &= 36^2+323^2\\ &= 13^2.7^2+13^2.24^2=91^2+312^2\\ &= 5^2.33^2+5^2.56^2=165^2+280^2\\ &= 5^2.63^2+5^2.16^2=315^2+80^2\\ &= (5.5)^2.5^2+(5.5)^2.12^2=125^2+300^2\\ &= (5.13)^2.3^2+(5.13)^2.4=195^2+260^2.\end{aligned}$$

II. *Note by E. P. Starke, Rutgers University.* The following is an obvious adaptation of a solution of problem no. 350 in the *National Mathematics Magazine*, Dec. 1940, p. 146, by G. W. Wishard, whose problem was to show that $5525=5^2\cdot 13\cdot 17$ has twenty-two representations as a sum of two squares.

If $c^2=a^2+b^2$, $ab\neq 0$, it is well known that we must have

$$a=2kxy,\qquad b=k(x^2-y^2),\qquad c=k(x^2+y^2),$$

where x and y have no common factor, one of them is even, and $x>y$. Further, a product $P=L\cdot M$ can be represented as a sum of two squares if each factor can, *viz.*

$$(1)\qquad (r^2+s^2)(u^2+v^2)=(ru+sv)^2+(rv-su)^2.$$

Conversely, every representation of P as a sum of two squares can be obtained from representations for L and M by use of (1). (See, *e.g.*, Carmichael, *Diophantine Analysis*, pp. 10, 24, ff.)

Now $325=5^2\cdot 13$, and $13=2^2+3^2$, $5=1^2+2^2$, $25=3^2+4^2$, whence various factorizations of 325 and repeated applications of (1) give the required seven representations as follows:

k	x^2+y^2	x	y	a	b
65	5	2	1	260	195
25	13	3	2	300	125
13	25	4	3	312	91
5	65	8	1	80	315
		7	4	280	165
1	325	17	6	204	253
		18	1	36	323

It is obvious from the theorems and methods employed that no smaller value of c than 325 can have the required property.

Solved also by E. D. Schell and the proposer: the latter did not give his method.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Arvid Jacobson has been appointed to an instructorship at Wayne University.

Professor R. D. James of the University of British Columbia has been granted a leave of absence to serve as operations analyst with the Third Air Force at Tampa, Florida.

Professor Mabel M. Heren of Knox College has retired as chairman of the department but will continue her teaching there. Professor R. C. Stephens succeeds her in the chairmanship.

Associate Professor W. H. Meyers of San Jose State College, California, has been appointed head of the department.

Professor Emeritus G. E. Robinson of the University of British Columbia died January 24, 1945.

SUMMER COURSES

The University of North Carolina: From July 2 to August 29 the following advanced courses in mathematics will be offered: By Professor Brauer: complex variable. By Professor Browne: history of mathematics. By Professor Henderson: differential equations. By Professor Lasley: analytic projective geometry. By Professor Winsor: college geometry. By Dr. Wong: mathematical theory of statistics.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

USAFI COURSES IN COLLEGE MATHEMATICS

Returning veterans who have studied courses under the auspices of the U. S. Armed Forces Institute are now entering colleges and universities in larger numbers. To assist departments of mathematics in planning a program for these men, the following information is given in regard to U.S.A.F.I. courses in mathematics. The descriptions here given are restricted essentially to courses which are regarded by the Institute as upon the college level. The statements of subject-matter are those provided by the U.S.A.F.I.

There are eleven correspondence courses in mathematics on the college level now offered directly to service personnel by the U.S.A.F.I. These courses are listed below. Credit equivalents are not given, of course, since the Institute does not grant or recommend credit.

C 712. *Plane Trigonometry.* Trigonometric functions; graphs; analysis; right and oblique triangle solution by natural functions and by logarithms; applications to surveying, physics, astronomy; inverse, exponential, and hyperbolic functions; trigonometric equations; De Moivre's Theorem. H 138 and H 139 (Beginning and Intermediate High School Algebra) are necessary preparation for this course. H 140 (Advanced High School Algebra) is desirable. Sixteen assignments. Text: *Plane and Spherical Trigonometry* (With Tables) by Nelson and Folley, 1943.

C 713. *College Algebra and Trigonometry.* Trigonometric functions; identities and equations; graphs; solution of right and oblique triangles; logarithms; inequalities; progressions; mathematical induction; theory of equations; probability; permutations and combinations. H 138, H 139, and H 140 (or equivalent courses) are required preparation for the course. H 143 and H 144 (Plane Geometry, two terms) are desirable. Thirty-two assignments. Texts: *College Algebra* by Rietz and Crathorne, 1939; *Brief Trigonometry* by E. A. Cameron, 1941.

C 714. *Plane Analytic Geometry.* Rectangular, oblique, and polar coordinates in the plane; the relation between a curve and its equation; properties of straight lines, circles, conic sections, and certain other plane curves. C 713 or H 138, H 139, H 140, and H 147 (High School Trigonometry) or C 712 (or equivalent courses) are necessary preparation. Sixteen assignments. Text: *Analytic Geometry* by C. E. Love, 1938.

C 715. *Descriptive Geometry.* The making and interpretation of the perspective drawings necessary in engineering, architecture and various fields of design. It considers the projections of points, lines, planes, and curved surfaces on

appropriate reference planes. Plane geometry and one semester of college mechanical drawing are required preparation for the course. Forty-five assignments. Text: *Descriptive Geometry* by F. H. Cherry, 1933.

C 716. *Spherical Trigonometry.* Solutions of right and oblique spherical triangles, and their applications. C 712 or any course in plane trigonometry is necessary preparation. Eight assignments. Text: *Plane and Spherical Trigonometry* (With Tables) by Rietz, Woods, and Reilly, 1942.

C 717. *Differential Calculus.* A study of the elements of calculus and their applications in finding areas, volumes, velocity and acceleration, motions of celestial bodies and atoms, flow of heat and electricity, probabilities, stresses of structural steel, compound interest, gas pressures, laws of growth and decay, and countless others. C 712 and C 714, or their full equivalent, is necessary preparation for the course. Forty assignments. Text: *Elements of the Differential and Integral Calculus* by Granville, Smith, and Longley, 1941.

C 718. *Integral Calculus.* Methods of integration with applications to areas of plane curves, volumes of solids of revolution, length of curve, areas of surfaces of revolution, moments of area; centroids of solids of revolution; and other problems in physics and mechanics. A knowledge of C 717, or similar background, is necessary preparation for this course. Forty assignments. Texts: *Elements of the Differential and Integral Calculus*, 1941 Edition, by Granville, Smith, and Longley; *Short Table of Integrals* by B. O. Peirce, 1929.

C 719. *Solid Analytic Geometry.* Continues the development and application of the methods of analytical geometry in the study of curves and surfaces, and the application of calculus to partial derivatives, multiple integrals, and infinite series. C 714 or equivalent is necessary preparation for this course. Sixteen assignments. Text: *Analytic Geometry* by C. E. Love, 1938.

C 724. *Differential Equations.* Fundamental types of ordinary differential equations, with applications to problems that are geometrical in nature and particularly to problems arising in physics and mechanics. A knowledge of C 717 and C 718, or equivalent, is necessary preparation for the course. Sixteen assignments. Text: *Differential Equations* by D. A. Murray, 1924.

C 725. *Engineering Mathematics—Part I.* A course in elementary analysis, covering the essentials of algebra, trigonometry, analytic geometry, and elementary calculus. One year each of algebra and geometry on the high school level are necessary, and an additional half-unit of algebra is desirable preparation for the course. Forty-six assignments. Text: *Introductory College Mathematics* by Milne and Davis, 1941.

C 726. *Engineering Mathematics—Part II.* A continuation of C 725. C 725 is necessary preparation. Twenty-two assignments. Text: *Introductory College Mathematics* by Milne and Davis, 1941.

The U.S.A.F.I. also makes available to service personnel a series of special reprints of standard textbooks to be used for group study or in off-duty classes. By requisition, an organization commander may obtain a quantity of these

books without charge. The following books in mathematics are now available.

EM 315. *College Algebra* by W. L. Hart, 1938. Covers the fundamentals of algebra on the college level.

EM 318. *Plane and Spherical Trigonometry with Tables* by Kells, Kern, and Bland, 1940. Covers the principles of both plane and spherical trigonometry with a unit on the slide rule, its construction and use.

EM 321. *Elements of Analytic Geometry* by C. E. Love, 1940. Covers the field of plane analytic geometry.

EM 324. *Elements of Differential and Integral Calculus* by Granville, Smith, and Longley, 1941. Covers fundamentals of differential and integral calculus.

EM 333. *Mathematics of Investment* by W. L. Hart. Provides an elementary study of the theory and application of annuities certain and of the mathematical aspects of life insurance.

EM 906. *A Course in the Slide Rule and Logarithms* by E. J. Hills, 1943. Explains the principles of the slide rule and logarithms, and their many practical uses. A polyphase slide rule is provided with each book. This is regarded as a high school course.

All other studies in mathematics now available to service men are definitely upon the elementary or secondary level, with the exception of two courses in trigonometry. These are listed below.

H 147. *Trigonometry.* The relations of the sides and angles of plane and spherical triangles; trigonometric functions; definitions; measurement and functions of angles; areas of plane figures; trigonometry and its usefulness in solving problems in navigation, mapping, artillery fire, surveying, engineering, and related fields. A knowledge of H 138 (Beginning Algebra) and H 139 (Intermediate Algebra), or similar background, is required. A knowledge of high school geometry is essential. Twenty assignments. Text: *Plane Trigonometry With Tables*, by Rosenbach, Whitman, and Moskovitz, 1941. This subject appears in the list of correspondence courses on the high school level.

Plane Trigonometry, Self-Teaching Manual EM 311. This manual is based on *Essentials of Trigonometry With Applications*, by Curtiss and Moulton, 1943. Covers trigonometric functions, reduction formulas, graphs, identities and equations, solution of triangles, and logarithms. A quantity of these self-teaching manuals may be obtained by organization commanders for use in regularly organized off-duty classes. An individual desiring to study the manual must enroll with the Institute, and make application to study the course.

RESEARCH BOARD FOR NATIONAL SECURITY

The recently announced Research Board for National Security is composed of ten Army officers of general rank and ten Naval officers of flag rank, and twenty civilians chosen from the fields of science, engineering, and industry. The executive committee of the Board is composed of Dr. Roger Adams, Head of the Department of Chemistry, University of Illinois; Dr. A. R. Dochez,

Professor of Experimental Medicine and Surgery, College of Physicians and Surgeons, Columbia University; Brigadier General W. A. Borden, Director of the New Developments Division, War Department Special Staff; Rear Admiral J. A. Furer, Coordinator of Research and Development, Navy Department; and Dr. K. T. Compton, President of the Massachusetts Institute of Technology, Chairman. The only mathematician on the Board is Dr. Oswald Veblen, Institute for Advanced Study, Princeton, New Jersey.

The new Board will be concerned with the advancement of science and technology in those directions which may have application in the conduct of future warfare. The ultimate establishment of the Research Board for National Security as an independent agency of government is now being considered by the Select Committee of the House on Post-War Military Policy. Temporarily the Board will continue as an agency of the National Academy of Sciences, thereby making available to it the resources and contacts of the National Research Council. For the duration of the war, it has been agreed that the new Board shall not engage in activities which are properly a function of the Office of Scientific Research and Development; this latter organization, of course, is an emergency wartime agency which will be disbanded shortly after the war.

EDUCATIONAL SURVEY OF THE ARMED FORCES

Recently the Information and Education Division of the Army Service Forces released Report No. B 121, pertaining to the educational demands of the members of the Armed Forces upon demobilization. The information which the report contains is based upon a survey of 20,000 white enlisted men. The following statements are quoted from the bulletin.

"Approximately 7 per cent of the white enlisted men in the Army are now definitely planning to return to full-time school after they leave the Army. On the basis of the current white enlisted strength of about $6\frac{3}{4}$ million, this represents close to a half million men, definitely planning to return to full-time school at the present time. Another 4 per cent of the men may possibly return to full-time school. Eighteen per cent of the men are planning to attend part-time school although one-third of these men would prefer to return to full-time school.

"Young men are more likely to be planning further education. Over ninety per cent of the men definitely planning to return to full-time school are less than twenty-five years old. Among men who want part-time education, less than two-thirds are under twenty-five.

"Over ninety per cent of the men who plan full-time school attendance have the formal educational requirements for college entrance. About two-thirds of them actually want college education. The remainder are primarily interested in trade and business schools. Over half the men who want part-time education are formally qualified for college courses, but three-fourths of them are planning trade or business courses.

"Probably not all the men now planning to go to full-time school will return, but others without definite plans at the present may decide to return. Seven

per cent is the best estimate at present of the number of men actually going back to full-time school. Factors such as the length of the war and favorable economic conditions may tend to decrease this proportion when demobilization actually occurs. Increasing knowledge of the educational provisions of the 'G.I. Bill' and less favorable conditions would tend to raise it."

A representative of the American Council on Education recently questioned fourteen education officers in the several overseas branches of the Armed Forces. These men indicated their belief that the educational interests of veterans would be primarily in technical and professional fields rather than in general education. Some officers made the qualification that younger veterans who had not completed their college education would be more interested in general education, while older veterans who had not been in college would be more interested in studies of a technical or professional nature. Two of the officers stated that service men had expressed special interest in courses in business administration and in engineering.

COMMITTEE ON THE TEACHING OF THE BASIC SCIENCES

On May 1, 1943, U. S. Commissioner of Education J. W. Studebaker asked for the formation of a committee to "canvass carefully the question of whether the basic sciences should not be included along with vocational education in a federally subsidized program. Such a subsidization would include, of course, the appropriate training of teachers for such courses." This resulted in the creation of a committee of four members representing the Commission on Teacher Education (Karl W. Bigelow), research in science teaching (R. J. Havighurst), the division of higher education of the United States Office of Education (F. J. Kelly), and the sciences (K. Lark-Horovitz, chairman of the committee).

After thorough consideration of the problems involved, it seemed apparent to the members of the committee that federal aid of some kind will be necessary to strengthen the teaching of the basic sciences. Accordingly, it has been proposed by the committee that existing legislation relating to vocational and technical education be amended, and future legislation be formulated to include provision, in addition to that now provided for such fields as vocational agriculture, home economics, trades and industries, and distributive occupations, for "the sciences basic thereto and (or including) mathematics." Essentially it is recommended that the Smith-Hughes and the George-Deen Acts be extended; these two acts provide for the cooperation between the federal government and the states in the promotion of certain types of vocational education.

According to a recent report of the Committee, the desired amendment, specifically as relating to the sections of the law having to do with the preparation and employment of teachers, would have the following effects:

"(1) As in the case of vocational agricultural and other vocational teachers, the standards which are set up by the state department of education in cooperation with the U. S. Office of Education will have to be met. This guarantees that certain minimum requirements would have to be fulfilled if federal aid

is to be obtained for the teaching of the sciences and mathematics as basic to agriculture, trades and industries, distributive occupations, and the proposed vocational-technical training.

"(2) Since the effectiveness of engineering, technical and vocational education and training is largely dependent on the soundness of preparatory education in mathematics, physics and chemistry, programs of teacher training under existing and future acts supplying federal support for teacher training should then include provision for teacher training in these fields."

The proposed amendment might well be part of a further modification of existing legislation that would also provide for:

"(1) Expansion of the present vocational and technical education program (now limited to work 'of less than college grade') to encompass activities 'of scholastic standard commonly associated with work done in high school and in the first two years beyond high school, but excluding that done as part of a regular four-year curriculum';

"(2) Authorization to the states to make use of and allocate support to the facilities of tax-exempt, though not publicly supported, universities, junior colleges, technical institutes, and the like, as well as to those publicly controlled;

"(3) Requirement that states—if they desire to participate—designate or create state boards to exercise control over the program within their boundaries, such boards to consist of not less than seven members so selected as to be representative of the various fields of interest for which the vocational and technical programs prepare."

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following fifty-one persons have been elected to membership on applications duly certified:

SILVIO AURORA. Student, Columbia Univ., New York, N. Y.

D. R. BEY, A.M.(Illinois) Asst. Prof., Illinois State Normal Univ., Normal, Ill.

W. H. BROTHERS, Jr., Ph.D.(Michigan) Prof., Talladega Coll., Talladega, Ala.

REV. W. F. BURNS, M.S.(Boston Coll.) Asso. Prof., Coll. of the Holy Cross, Worcester, Mass.

JOSEPHINE H. CHANLER, Ph.D.(Illinois) Asst. Prof., Univ. of Illinois, Urbana, Ill.

WAY MING CHEN, B.S.(Nat'l. Southwest Asso. Universities, Kunming, China) Teaching Asst., Univ. of California, Berkeley, Calif.

SISTER MARIA LOYOLA CONLAN, A.M.(Fordham) Asso. Prof., Coll. of Mount St. Vincent, New York, N. Y.

W. R. DOLL, A.B.(Columbia Coll.) Engr., Radio Station WKNE, Keene, N. H.

MARY A. ENGLISH. Student, Spelman Coll., Atlanta, Ga.

E. J. EVENSEN. Actuarial Section, Metropolitan Life Ins. Co., Pacific Coast Head Office. On leave. T/4, U. S. Army.

HANS FRIED, Ph.D.(Vienna) Lecturer, Swarthmore Coll., Swarthmore, Pa.

A. H. S. GILLSON, M.A.(Cambridge) Prof., McGill Univ., Montreal, P.Q., Canada

E. S. GRABLE, A.M.(Washington and Jefferson) Asst. Prof., Univ. of Richmond, Richmond, Va.

W. C. GRIFFITH, A.M.(Oregon) Instr., DePauw Univ., Greencastle, Ind.

L. B. GUELPA, JR., A.M.(New York Univ.) Lt., U.S.N.R. Asso. Educational Officer in Math., U. S. Merchant Marine Acad., Kings Point, N. Y.

MRS. MARIAN S. GYSLAND, A.B.(Colorado) Teaching Asst., Univ. of California, Berkeley, Calif.

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THE COORDINATING COMMITTEE

Some Sections of the Association have active committees devoted to the furthering of the interests of sound education in general and of mathematics in particular. Some of these committees have real accomplishment to their credit as a result of advising local educational groups, state boards of education, and legislative committees. They generally find that their advice is treated with respect in such matters as curriculum changes and teacher training.

As post-war educational problems press upon us, it is probable that most if not all of the Sections will find it desirable to establish such committees, possibly in conjunction with other organizations whose aims and ideals are similar to our own. Since education is still a local matter in this country, it is only through local committees, preferably at least one committee for each State, that progress can be made. Not only do the problems vary from place to place, but persons in authority are amenable only to committees of local origin.

With these thoughts in mind, the Board of Governors of the Association has authorized the appointment of a Coordinating Committee whose function it shall be to keep in close touch with all educational movements in the United States and Canada, and to make the information thus acquired available to all committees of the Association and its Sections who can profit by this information. It is hoped that in return all Sectional committees will make contact with the Coordinating Committee and report the problems which are facing them, their plans to meet these problems, and their successes or failures. The Coordinating Committee may be able to supply the local committees with reprints of pertinent articles, or at least with references to them. In any case the local committees should be able to work with more enthusiasm and confidence if they are fortified with exact information about what is going on in other places.

The Association is fortunate in having secured the services of an able committee. The chairman is Professor C. V. Newsom of Oberlin College, Oberlin, Ohio, and the other two members are Professor M. S. Knebelman of the State College of Washington at Pullman, and Professor W. V. Parker of the Louisiana State University at Baton Rouge.

C. C. MacDuffee, *President*

THE FALL MEETING OF THE INDIANA SECTION

The twenty-second annual meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, Indiana, on Friday, November 10, 1944, in conjunction with the meeting of the Indiana Academy of Science. Professor Emil Artin, Chairman of the Section, presided at the morning session, and Professor P. M. Pepper, Chairman of the Mathematics Section of the Academy, presided at the afternoon session.

Thirty-two persons registered at the meeting, including the following eighteen members of the Association: Emil Artin, Juna Lutz Beal, G. E. Carscallen, J. E. Dotterer, W. E. Edington, P. D. Edwards, G. H. Graves, Cora B. Hennel,

M. W. Keller, Mark Lotkin, H. A. Meyer, P. M. Pepper, J. C. Polley, D. H. Porter, J. W. Wiley, K. P. Williams, H. E. Wolfe, Sister Gertrude Marie Zieroff.

At the business meeting the following officers were elected for the next year: Chairman, Juna Lutz Beal, Butler University; Secretary, M. W. Keller, Purdue University.

The following papers were presented:

1. *The great mathematics books in the college curriculum*, by Sister Gertrude Marie Zieroff, O. S. F., Marian College.

In this paper the speaker evaluated the method of learning college mathematics directly from the great mathematics classics. The technique employed for this type of instruction at St. John's College was described. Representative classics were compared with college text-books. The feasibility of using mathematics classics for collateral reading as part of the regular course, for honors courses, and for seminars, was discussed.

2. *On certain recursion inequalities with applications*, preliminary report, by Professor P. M. Pepper, University of Notre Dame.

Professor Pepper dealt with certain problems relating to a switchboard with n terminals, and with wires connecting the terminals in pairs. He considered the determination of the greatest number of cross-connections which can be made without there being somewhere three terminals each two of which are joined by a wire. Knowing the answer to this question, one may ask for a distribution of the maximum number of wires in such a way as to form no triangles (*i.e.*, no three terminals each two of which are connected). In the study of such questions he was led to the consideration of the following auxiliary problem: Let a, b, c, and u_0 be given integers with $a \geqq 0$; find a simple formula for u_n in terms of a, b, c, u_0 and n if u_n is the least integer satisfying the inequality

$$u_n \geqq [(n + a + c)u_{n-1} - (n + b)]/(n - a), \quad n = 1, 2, 3, \cdots.$$

The present paper contains a solution of the first two problems and the solution of a two-parameter family of the recursion inequalities with restricted u_0.

3. *What are we teaching mathematics for?* by Professor G. H. Graves, Purdue University.

In this paper the author pointed out that an essential feature of mathematics is the development of the implications of a set of assumptions. He stated that one of the prime purposes of mathematics is to convey an appreciation of this viewpoint, and that we should examine whether this objective is not in danger of being submerged in the many applications of mathematical processes. It was affirmed that there is a higher practicality to the grasp of a technique for drawing conclusions from a set of data, to the assembling and criticizing of data, and to the investigation of the assumptions on which the argument proceeds, than to any particular results of this process, however important these may be.

4. *Some illustrations of the Hamilton-Jacobi theory*, by Professor K. P. Williams, Indiana University.

Professor Williams explained the importance of the Hamilton-Jacobi differential equation theorems as they apply to planetary theories. An example, in which all integrations could be carried through, was given to show how the solution of one Hamilton system could be made to furnish the solution of a modified system.

5. *Some remarks on final grades in freshman mathematics*, by Professors M. W. Keller and H. S. F. Jonah, Purdue University.

Some data was presented which indicates from a preliminary study that certain tendencies exist in giving final grades when ordinary final examinations are given, when no examinations are given, and when uniform objective final examinations are given.

6. *Determinants*, by Professor Emil Artin, Indiana University.

Professor Artin presented a new set of axioms for determinants. The axioms were: (1) linearity and homogeneity as a function of the columns of a matrix; (2) the vanishing, if two adjacent cloumns are equal; (3) the value one in case of the unit matrix. From these axioms the speaker led very quickly to all important properties of determinants without introducing more than the elementary notions of permutations.

M. W. KELLER, *Secretary*

CALENDAR OF FUTURE MEETINGS

The Office of Defense Transportation has refused permission for our previously announced meeting at Montreal, June 23–25, 1945, and this meeting has therefore been cancelled.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, October 19, 1945
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May, 1945
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in November 1944 to Professor R. H. Cameron for his paper, "Some introductory exercises in the manipulation of Fourier transforms," published in the *National Mathematics Magazine*, vol. 15 (1941), pp. 331-356.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

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